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Weakly Pulse-Coupled Oscillators: Heterogeneous Delays Lead to Homogeneous Phase

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***Abstract*— This paper studies the effect of heterogenous delays in networks of weakly pulse-coupled identical oscillators. We develop a new framework to study them by constructing a non-delayed phase model that is equivalent to the original one in the continuum limit. Using existing results for non-delayed phase-coupled oscillators we analyze the delayed system and show how its stability properties depend on the delay distribution. In particular, we show that in some scenarios, heterogeneity, i.e. wider delay distribution, can help reach in-phase synchronization.**

I. INTRODUCTION

The model of coupled oscillators has been widely used in several disciplines ranging from biology [28], [22], [1], [7], [29] and chemistry [13], [31] to engineering [9], [27] and physics [18], [2]. It characterizes local interactions among oscillators and can generate complex dynamical behavior including stable and unstable equilibria [4], oscillations [15], and even chaos [23].

One particularly interesting question is whether the coupled oscillators can synchronize in phase in the long run. There has been active research work regarding this question (see for e.g., [19], [16], [20], [21]). However, they typically assume zero delays among the oscillators and therefore are not satisfactory for many applications.

In this paper, we develop a new framework to study weakly pulse-coupled oscillators with delays by constructing an equivalent non-delayed system that has the same behavior as the original one in the continuum limit. We then further use this result to show that heterogeneous delays can help reach synchronization, which is a bit counterintuitive and significantly generalizes previous related studies [11], [25], [8].

The rest of the paper is organized as follows. In Section II, we briefly introduce the model and state previous pertinent results. We then study the effect of delays in a network of two oscillators in Section III-A and build the non-delayed approximation for a large population of oscillators in Section III-B. In Section IV we use the new approximation to analyze stability and provide numerical results to verify what our theory predicts. Conclusions are presented in Section V.

II. PRELIMINARIES

In the canonical model of pulse-coupled oscillators [11], each oscillator i is represented as a point θ_i in the unit circle $\mathbb{S}^1$ that moves with constant speed, i.e.

$$\dot{\theta}_i = \omega \quad \forall i \in \mathcal{N},$$

where $\omega = \frac{2\pi}{T}$ is the natural frequency of oscillation and $\mathcal{N}$ is the set of all oscillators whose cardinality is N.

An oscillator $j \in \mathcal{N}$ sends out a pulse whenever it crosses zero ($\theta_j = 0$). When oscillator i receives a pulse, it will change its position from θ_i to $\theta_i + \varepsilon\kappa(\theta_i)$. The function κ represents how the actions of other oscillators affect i and the scalar $\varepsilon > 0$ is a measure of the coupling strength. These jumps can be modeled by Dirac's delta functions, δ, satisfying $\delta(t) = 0\ \forall t \neq 0$, $\delta(0) = +\infty$, and $\int \delta(s)ds = 1$. Using the δ function, the coupled dynamics is represented by

$$\dot{\theta}_i(t) = \omega + \varepsilon\omega \sum_{j\in\mathcal{N}_i} \kappa(\theta_i(t))\delta(\theta_j(t-\eta_{ij})), \qquad (1)$$

where $\eta_{ij} > 0$ is the propagation delay between i and j ($\eta_{ij} = \eta_{ji}$), and $\mathcal{N}_i$ is the set of i's neighbors. The factor of ω in the sum is needed to keep the size of the jump within $\varepsilon\kappa(\theta_i)$. This is because $\theta_j(t)$ behaves like ωt when crosses zero and therefore the jump produced by $\delta(\theta_j(t))$ is of size $\int \delta(\theta_j(t))dt = \omega^{-1}$.

This pulse-like interaction between oscillators was first introduced by Peskin [22] in 1975 as a model of the pacemaker cells of the heart, although the canonic form did not appear in the literature until 1999 [11]. The coupling function κ is usually classified based on its sign; if $\kappa > 0$, the coupling is *excitatory* and if $\kappa < 0$, then it is called *inhibitory* coupling. This classification is based on models of biological oscillatory networks, but is not sufficient even to characterize the system's qualitative behavior and usually a first order derivative condition is needed to obtain desired synchronization.

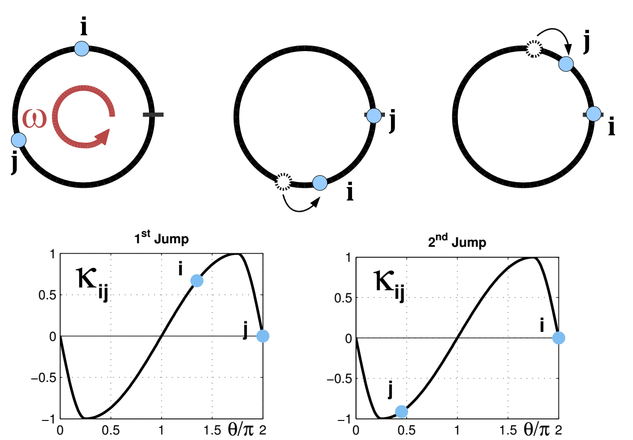

Fig. 1. Pulse-coupled oscillators with attractive coupling: After the two oscillators fire the phases get closer

Here we introduce a different criteria that takes into

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account the qualitative behavior of the system. After one period, if in the *absence of delay* the net effect of the mutual jumps brings a pair of oscillators closer, we call it *attractive* coupling. If the oscillators are brought further apart, we call it *repulsive*. This can be achieved for instance if $\kappa(\theta) \leq 0$ for $\theta \in [0, \pi)$ and $\kappa(\theta) \geq 0$ for $\theta \in [\pi, 2\pi)$. See Figure 1 for an illustration of an attractive coupling κ and its effect on the relative phases. Notice that this new classification only refers to the net effect that κ produces *without delay*. When delay is included, an attractive κ can produce repulsive net effect. This behavior is further discussed in Section III-A.

When there are only two oscillators without propagation delay, the behavior of the system is easy to predict. When the coupling is attractive, unless they start with a difference of exactly half a period, both oscillators will bring their phases closer after every period, as in Figure 1, until they eventually synchronize in-phase, i.e. both phases achieve consensus. When the coupling is repulsive, exactly the opposite behavior occurs; the phases will end up as far as possible, i.e. within a distance of π (anti-phase). However, when the number of oscillators increases, there are more and more possible different outcomes besides these two [4].

Thus, predicting whether the system reaches in-phase synchronization is not an easy task. The problem was solved for the complete graph case by Mirollo and Strogatz in 1990 [19] by showing that if $\kappa(\theta)$ is strictly increasing (which resembles attractive coupling), then for almost every initial condition, the system can synchronize in phase in the long run.

The analysis in [19] strongly depends on the assumptions of complete graph and zero delays among oscillators. When the graph is no longer complete, each oscillator receives a different firing pattern which makes the proof in [19] no longer valid. On the other hand, when delays among oscillators are introduced, which is necessary for many interesting cases in practice, the analysis becomes intractable. Even for the case of two oscillators, the number of possibilities to be considered is large [5], [6].

In this paper, we assume that the coupling strength is weak, i.e. $1 \gg \varepsilon > 0$, such that the effect of the jumps originated by each neighbor can be approximated by their average [10]. This gives a natural continuous approximation of the previous dynamics,

$$\dot{\phi}_i = \omega + \varepsilon \sum_{j \in \mathcal{N}_i} H(\phi_j - \phi_i - \psi_{ij}) \quad \forall i \in \mathcal{N} \qquad (2)$$

where now each neighbor $j \in \mathcal{N}_i$ changes i's **speed** by an amount depending on their phase difference. The function

$$H(\theta) = \frac{\omega}{2\pi}\kappa(-\theta) \qquad (3)$$

is 2π-periodic and inherits the role of κ in the previous model. The phase lag $\psi_{ij} = \omega\eta_{ij}$ represents the distance that j can travel along the unit circle in the delay time η_{ij}.

Remark 1 *Equation* (2) *represents a system of continuously phase-coupled oscillators with phase lags. This model should not be confused with the model of delayed phase-coupled oscillators, see e.g. [24], [30], [21]. Although their behaviors are similar when the coupling is weak, for strong coupling their dynamics can vary significantly.*

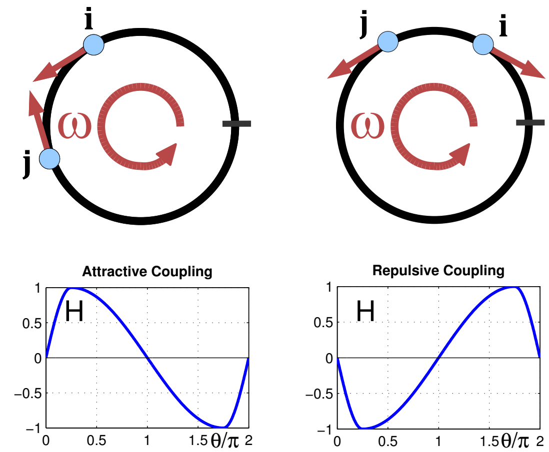

Fig. 2. Phase-coupled oscillators with attractive and repulsive coupling: The arrows represent the speed change produced by the other oscillator; if the pointing direction is counter clockwise, the oscillator speeds up, and otherwise it slows down.

Remark 2 *From now on we will concentrate on* (2) *with the understanding that any convergence result derived also holds for the original weakly pulse-coupled model. Therefore, we will treat both models as representative of the dynamics of a weakly pulse-coupled network of oscillators and in this sense the phase lags will also be interpreted as delay. For more details about this approximation we refer the reader to [10] and [11].*

Using (3) we can translate the attractive/repulsive coupling classification in terms of H. Thus, a coupling is attractive if the mutual changes in speed bring the oscillators closer and repulsive if they are repelled from each other. Figure 2 shows a typical attractive and repulsive H. Notice that in order to produce the same effect (either attractive or repulsive) κ and H should be mirrored.

Once delay is introduced to the system, the problem becomes fundamentally harder. The reception of a pulse gives no useful information about the relative phase difference $\Delta\phi_{ij} = \phi_j - \phi_i$ between the two interacting oscillators. Before, at the exact moment when i received a pulse from j, ϕ_j was zero and the phase difference was estimated locally by i as $\Delta\phi_{ij} = -\phi_i$. However, now when i receives the pulse, the difference becomes $\Delta\phi_{ij} = -\phi_i - \psi_{ij}$. Therefore, the delay propagation acts as an error introduced to the phase difference measurement and unless some information is known about this error, it is not possible to predict the behavior. Moreover, as will see in the next section, slight changes in the distribution can produce nonintuitive behaviors.

III. Effect of Delay

In this section we show how propagation delays affects the dynamics of a network of weakly pulse-coupled oscillators. We will assume complete graph to simplify notation and exposition although the results can be extended for a boarder class of densely connected networks.

A. Two Oscillators

Suppose first that there are only two oscillators, $\mathcal{N} = \{1, 2\}$, with coupling function $H(\phi_j - \phi_i - \psi_{ij}) = K \sin(\phi_j - \phi_i - \psi_{ij})$, i.e., the classical Kuramoto model [14]. Then (2) becomes

$$\dot{\phi}_i = \omega + K \sin(\phi_j - \phi_i - \psi) \qquad i, j \in \{1, 2\},\, j \neq i. \quad (4)$$

For $K > 0$, the coupling is **attractive**. Hence when $\psi = 0$, unless the initial phase difference is π, both oscillators will bring their phases closer until they synchronize in-phase. Similarly, if $K < 0$, the coupling is **repulsive** and the oscillators will move towards the anti-phase state. Therefore, when there is no delay, one would tend to use $K > 0$ in order to synchronize in-phase.

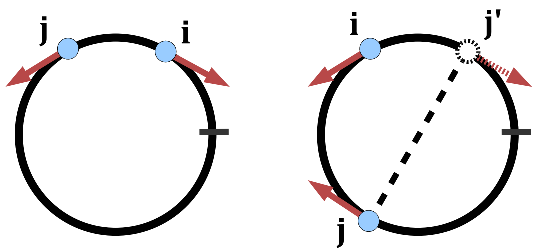

Fig. 3. Repulsive coupling without delay vs. Repulsive coupling with delay of π: The delay can produce an attractive net effect even with a repulsive H (or equivalently κ).

What is interesting here is the effect of the delay. For instance, when $\psi = \pi$, a simple change of variable $\bar{\phi}_1 = \phi_1$, $\bar{\phi}_2 = \phi_2 - \pi$ transforms (4) into

$$\dot{\bar{\phi}}_i = \omega + K \sin(\bar{\phi}_j - \bar{\phi}_i) \qquad i, j \in \{1, 2\},\, j \neq i. \quad (5)$$

Then from the previous discussion, by using attractive coupling ($K > 0$) the system will tend to align $\bar{\phi}_1$ with $\bar{\phi}_2$. However, this implies that the phase difference between ϕ_1 and ϕ_2 is π. On the other hand, when repulsive coupling ($K < 0$) is used, the $\bar{\phi}_i$ variables reach the anti-phase configuration, which produces in turn in-phase synchronization for the original system.

The intuition behind this behavior is that performing repulsive actions over another oscillator whose phase is within π from where it is supposed to be has the net effect of bringing both oscillators closer instead of further apart. See Figure 3 for an illustration of this effect.

B. Large Number of Oscillators

We now generalize the above intuition to a network of large number of oscillators. This is a challenging task since the heterogeneity in the propagation delays makes impossible to extend previous analysis. We shall build on existing arguments such as mean field approximation [15] and Lyapunov stability theory [20], [12] while looking at the problem from a different perspective.

Consider the case where the coupling between oscillators is all to all ($\mathcal{N}_i = \mathcal{N} \backslash \{i\}$, $\forall i \in \mathcal{N}$) and the phase lags ψ_{ij} are randomly and independently chosen from the same distribution with probability density $g(\psi)$. By letting $N \to +\infty$ and $\varepsilon \to 0$ while keeping $\varepsilon N =: \bar{\varepsilon}$ constant, (2) becomes

$$v(\phi, t) := \omega + \bar{\varepsilon} \int_{-\pi}^{\pi} \int_{0}^{+\infty} H(\sigma - \phi - \psi) g(\psi) \rho(\sigma, t) d\psi d\sigma, \quad (6)$$

where $\rho(\phi, t)$ is a time-variant normalized phase distribution that keeps track of the fraction of oscillators with phase ϕ at time t, and $v(\phi, t)$ is the velocity field that expresses the net force that the whole population applies to a given oscillator with phase ϕ at time t. Since the number of oscillators is preserved at any time, the evolution of $\rho(\phi, t)$ is governed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \phi}(\rho v) = 0 \quad (7)$$

with the boundary conditions $\rho(0, t) \equiv \rho(2\pi, t)$.

Equations (6)-(7) are not analytically solvable in general. Here we propose a new perspective that is inspired in the following new observation.

Theorem 1 *Mean Field Approximation*
Let ψ_{ij} be independent and identically distributed random variables with probability density function $g(\psi)$. Then, there is a non-delayed system of the form

$$\dot{\phi}_i = \omega + \varepsilon \sum_{j \in \mathcal{N}_i} F(\phi_j - \phi_i), \quad (8)$$

where

$$F(\theta) = H * g(\theta) = \int_{0}^{+\infty} H(\theta - \psi) g(\psi) d\psi \quad (9)$$

is the convolution between H and g such that (2) *and* (8) *have the same continuum limit.*

Proof: By the same reasoning of (6) it is easy to see that the limiting velocity field of (8) is

$$\begin{aligned}
v_F(\phi, t) &= \omega + \bar{\varepsilon} \int_{0}^{2\pi} F(\sigma - \phi) \rho(\sigma, t) d\sigma \\
&= \omega + \bar{\varepsilon} \int_{0}^{2\pi} \left(\int_{0}^{+\infty} H((\sigma - \phi) - \psi) g(\psi) d\psi \right) \rho(\sigma, t) d\sigma \\
&= \omega + \bar{\varepsilon} \int_{0}^{2\pi} \int_{0}^{+\infty} H(\sigma - \phi - \psi) g(\psi) \rho(\sigma, t) d\psi d\sigma \\
&= v(\phi, t)
\end{aligned}$$

where in the first step we used (9) and in the third (6). Thus both systems produce the same velocity field in the limit and therefore behave identically. ■

Remark 3 *Although* (8) *is quite different from* (2)*, Theorem 1 states that both systems behave exactly the same in the continuum limit. Therefore, as N grows,* (8) *starts to become a good approximation of* (2) *and therefore can be analyzed to understand the behavior of* (2)*, and* (1).

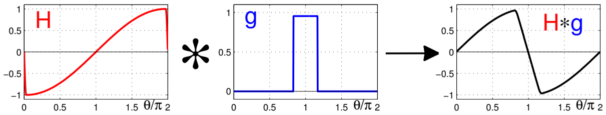

Fig. 4. Effect of delay in coupling shape: The original function H produces repulsive coupling, whereas the corresponding F is attractive

Figure 4 shows how, as in the case of two oscillators, the underlying delays (in this case the delay distribution) determine what type of coupling (attractive or repulsive) produces synchronization. The original function H produces repulsive coupling, whereas the corresponding F is attractive. In fact, as we will soon see, the distribution of delay not only can qualitatively affect the type of coupling but also can change the stability of certain phase-locked limit cycles.

IV. Stability Analysis and Numerical Results

In this section we study two examples to illustrate how this new approximation can provide significant information about performance and stability of the original system. We also provide numerical simulations to verify our predictions.

A. Kuramoto Model

We start by studying an example in the literature [26] to demonstrate how we can use the previous equivalent non-delayed formulation to provide a better understanding of systems of weakly pulse-coupled oscillators with delays. We assume $H(\theta) = K\sin(\theta)$. In this case $F(\theta)$ can be easily calculated:

$$\begin{aligned} F(\theta) &= \int_0^{+\infty} K\sin(\theta-\psi)g(\psi)d\psi \\ &= K\int_0^{+\infty} \Im\left[e^{i(\theta-\psi)}g(\psi)\right]d\psi \\ &= K\Im\left[e^{i\theta}\int_0^{+\infty} e^{-i\psi}g(\psi)d\psi\right] \\ &= K\Im\left[e^{i\theta}Ce^{-i\xi}\right] \\ &= KC\sin(\theta-\xi) \end{aligned}$$

Here $\Im$ is the imaginary part of a complex number, i.e. $\Im[a+ib]=b$. The first and second step follow from linearity of the integral, and the third from defining $C>0$ and ξ using the identity

$$Ce^{i\xi} = \int_0^{+\infty} e^{i\psi}g(\psi)d\psi.$$

This complex number ($Ce^{i\xi}$), usually called "order parameter", provides a measure of how the phase-lags are distributed within the unit circle. It can also be interpreted as the center of mass of the lags ψ_{ij} when they are thought of as points ($e^{i\psi_{ij}}$) within the unit circle $\mathbb{S}^1$. Thus, when $C \approx 1$, the lags are mostly concentrated around ξ, and when $C\approx 0$, they are distributed such that $\sum_{ij} e^{i\psi_{ij}} \approx 0$.

In this example, (8) becomes

$$\dot{\phi}_i = \omega + \varepsilon KC\sum_{j\in\mathcal{N}_i}\sin(\phi_j-\phi_i-\xi). \tag{10}$$

Here we see how the distribution of $g(\psi)$ has a direct effect on the dynamics. For example, when the delays are heterogeneous enough such that $C\approx 0$, the coupling term disappears and makes synchronization impossible. A complete study of the system under the context of superconducting Josephson arrays was performed in [26] for the complete graph topology. There the authors characterized the condition for in-phase synchronization in terms of K and $Ce^{i\xi}$. More precisely, when $KCe^{i\xi}$ is on the right half of the complex plane ($KC\cos(\xi)>0$), the system almost always synchronizes. However, when $KCe^{i\xi}$ is on the left half of the complex plane ($KC\cos(\xi)<0$), the system moves towards an incoherent state where all the phases spread around the unit circle such that its order parameter, i.e. $\frac{1}{N}\sum_{l=1}^{N} e^{i\phi_l}$, becomes zero.

Another way to interpret the parameter ξ is as if every signal were delayed by the same amount. Then, the previous conditions are akin to the ones discussed for two oscillators with delay, where we showed that if $\xi=\pi$ ($\cos(\xi)=-1$) then $K<0$ (repulsive coupling) produces synchronization, whereas when $\xi=0$ ($\cos(\xi)=1$), $K>0$ is the one that synchronizes.

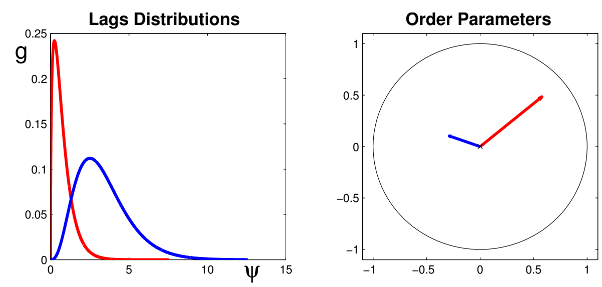

Fig. 5. Delay distributions and their order parameters $Ce^{i\xi}$

We now provide simulation results to illustrate how (10) becomes a good approximation of (2) when N is large enough. We simulate the original repulsive ($K<0$) sine-coupled system with heterogeneous delays and its corresponding approximation (10). Two different delay distributions, depicted in Figure 5, were selected such that their corresponding order parameter lie in different half-planes.

The same simulation is repeated for $N=5,10,50$. Figure 6 shows that when N is small, the phases' order parameter of the delayed system (in red/blue) draw a trajectory which is completely different with respect to its approximation (in green). However, as N grows, in both cases the trajectories become closer and closer. Since $K<0$, the trajectory of the system with wider distribution ($C\cos\xi<0$) drives the order parameter towards the boundary of the circle, i.e., **heterogeneous delays lead to homogeneous phase**.

B. Effect of Heterogeneous Delays

We now explain a more subtle effect that heterogeneous delays can produce. Consider the system in (8) where F after the convolution is odd and continuously differentiable. It is known that under such conditions, all of the oscillators eventually end up running at the same speed ω [3]. The

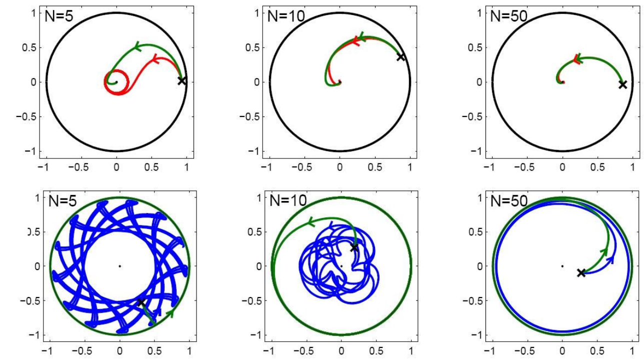

Fig. 6. Repulsive sine coupling with the heterogeneous delays distributed according to Fig. 5: As N grows, the trajectories of (10) (in green) become closer to the original ones (in red and blue).

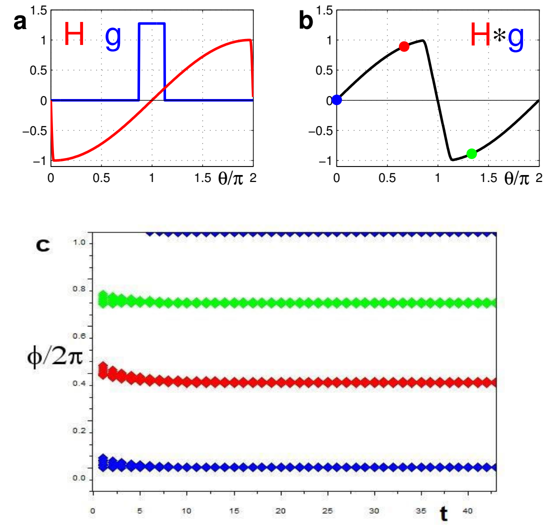

Fig. 7. Pulse-coupled oscillators with delay: Stable equilibrium

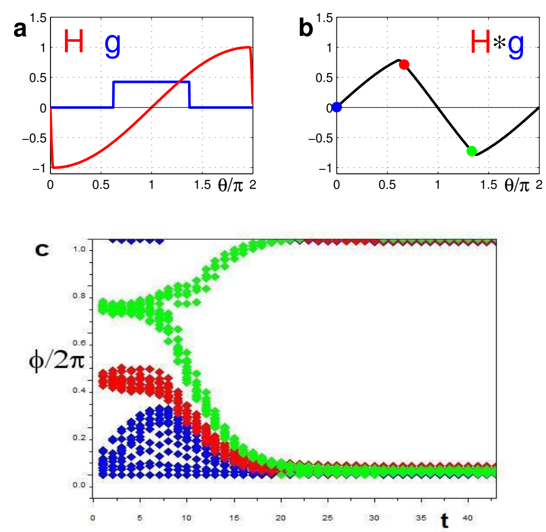

Fig. 8. Pulse-coupled oscillators with delay: Unstable equilibrium

complication is that the relative phases might not be all aligned in phase.

Instead, they might form several different types of constellations along the unit circle [4] such that the sum $\sum_{j \in \mathcal{N}_i} F(\phi_j - \phi_i)$ cancels $\forall i$. Whether these solutions are stable or not can be addressed by linearizing around the equilibria ϕ^* of the system

$$\dot{\phi}_i = \sum_{ij \in \mathcal{N}_i} F(\phi_j - \phi_i).$$

A sufficient condition for the instability [17] of such solutions is the existence of a cut $\mathcal{K}$ of the network such that

$$\sum_{ij \in \mathcal{K}} F'(\phi_j^* - \phi_i^*) < 0.$$

One way to ensure synchronization is to use a function F that guarantees that any other solution, except the in-phase one, is unstable.

Although the condition is for non-delayed phase-coupled oscillators, the result of this paper allows us to translate it for systems of pulse coupled oscillators with delay. Since F is the convolution of the coupling function H and the delay distribution function g, we can obtain $F'(\phi_j^* - \phi_i^*) < 0$, even when $H'(\phi_j^* - \phi_i^*) > 0$. This usually occurs when the convolution widens the region with negative slope of F. See Figure 4 for an illustration of this phenomenon.

Figures 7 and 8 show two simulation setups of 45 oscillators pulse-coupled all to all. The initial state is close to a phase locked configuration formed of three equidistant clusters of 15 oscillators each. The shape of the coupling function H and the phase lags (delay) distributions are shown in **a**; the corresponding κ used in the simulation can be inferred using (3). While H (κ) is maintained unchanged between both simulations, the distribution g does change. Thus, the corresponding $F = H * g$ also changes as it can be seen in **b**; the *blue, red*, and *green* dots correspond to the speed change induced in an oscillator within the blue cluster by oscillators of each cluster. Since all clusters have the same number of oscillators, the net effect is zero. In **c** the time evolutions of the oscillators' phases relative to the phase of an oscillator of the *blue* cluster are shown. Although the initial conditions are exactly the same, the wider delay distribution on Figure 8 produces negative slope on the *red* and *green* points of **b**, which destabilizes the clusters and drives the oscillators toward in-phase synchrony.

Finally, we simulate the same scenario as in Figures 7 and 8 but now changing N and the standard deviation, i.e. the lags distribution width. Figure 9 shows the computation of the synchronization probability vs. standard deviation. The dashed line denotes the minimum value that destabilizes the equivalent system. As N grows, the distribution shape becomes closer to a step, which is the expected shape in the limit.

V. Conclusions

We have studied networks of weakly pulse-coupled oscillators with delays. In the continuum limit of a large number of oscillators, we construct an equivalent non-delayed system

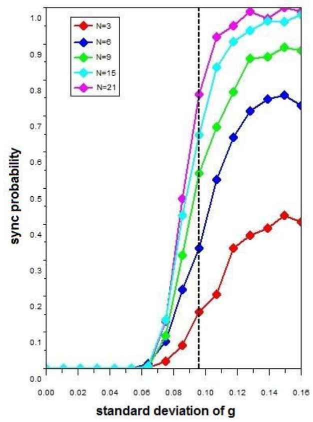

Fig. 9. Pulse-coupled oscillators with delay: Synchronization probability

that has the same dynamical behavior as the original one with delays. By analyzing this non-delayed system, we are able to examine the dynamics of the original system and show how delays affect important system behaviors such as synchronization. In particular, we predict and demonstrate that repulsive coupling can produce synchronization when the delays among oscillators are sufficiently heterogeneous.

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