Two-Hop Interference Channels: Impact of Linear Time-Varying Schemes

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Abstract—We consider the two-hop interference channel (IC) with constant channel coefficients, which consists of two source-destination pairs, separated by two relays. We analyze the achievable degrees of freedom (DoF) of such network when relays are restricted to perform scalar amplify-forward (AF) operations, with possibly time-varying coefficients. We show that, somewhat surprisingly, by providing the flexibility of choosing time-varying AF coefficients at the relays, it is possible to achieve 4/3 sum-DoF. We also develop a novel outerbound that matches time-varying AF coefficients at the relays, hence characterizing the sum-DoF of two-hop interference channels with time-varying AF relaying strategies.

I. INTRODUCTION

Multi-hopping is typically viewed as an effective approach to extend the coverage range of wireless networks, by bridging the gap between the sources and destinations via relays. However, it has also the potential to significantly impact network capacity by enabling new interference management techniques (see, e.g., [1]–[3]). In particular, from the degrees of freedom (DoF) perspective that is the focus of this paper, authors in [4] considered a two-hop interference channel (IC) consisting of two sources, two relays, and two destinations, and by introducing a new scheme called aligned-interference-neutralization, they showed that the sum-DoF of this network is 2 (i.e., twice the sum-DoF of a single-hop IC). Also, more recently in [5], two-hop interference networks with K sources, K relays and K destinations have been considered, and by developing a new scheme named aligned-network-diagonalization, it has been shown that relays have the potential to asymptotically cancel the interference between all source-destination pairs, hence achieving K-sum DoF (i.e., the cut-set bound).

While the aforementioned results essentially demonstrate that significant DoF gains can be achieved by carefully designing the interference management strategies in multi-hop interference networks, they often require complicated relaying strategies (such as, utilizing rational dimensions for neutralizing the interference when the channels are not time-varying). In this paper, we take a complementary approach and ask how much of these DoF gains can be realized if we limit the operation of relays to simple scalar linear strategies?

We focus on two-hop interference channels with constant channel coefficients (i.e., slow fading), and assume that the relays are only allowed to perform scalar amplify-forward (AF) operations, with possibly time-varying AF coefficients. It is easy to see that if AF coefficients of the relays remain constant during the course of the scheme, then the problem will induce to a single-hop IC, in which the sum-DoF is at most 1. However, we show that, somewhat surprisingly, by providing the flexibility of choosing time-varying AF coefficients at the relays, a sum-DoF of 4/3 is achievable.

The key idea behind the achievability strategy is that the flexibility of choosing the relay AF factors allows canceling, in any specific time slot, one source signal from one destination. So, we use this flexibility to guarantee that, for each destination, at most one third of its received symbols are distinct interference symbols, which allows it to achieve 2/3 DoF.

To derive the outerbound, we break the end-to-end mutual information achieved by any scheme into five different groups, based on five distinct states that scalar linear schemes can create at each time-step. We then proceed to prove three outerbounds that effectively capture the tension between these groups. Analyzing the three bounds yields that the sum-DoF is upper bounded by 4/3 almost surely.

II. PROBLEM SETTING & MAIN RESULT

As illustrated in Figure 1, we consider the two-hop IC, consisting of two sources, two relays, and two destinations.

![Fig. 1. Two-hop IC.](image)

We denote the two sources by $s_1$ and $s_2$, the two relays by $u$ and $v$, and the destinations by $d_1$ and $d_2$. Each source $s_i$ has a message $W_i$ intended for $d_i$ ($i \in \{1, 2\}$), and $W_1 \perp W_2$.

Let $H_1 = \begin{bmatrix} h_{s_1,u} & h_{s_2,u} \\ h_{s_1,v} & h_{s_2,v} \end{bmatrix}$ and $H_2 = \begin{bmatrix} h_{u,d_1} & h_{v,d_1} \\ h_{u,d_2} & h_{v,d_2} \end{bmatrix}$ be the channels of the first and second hop, respectively. We assume that the channel gains are real-valued and drawn from a continuous distribution, fixed during the course of communication, and known at all nodes.

The transmit signal of $s_i$ and relay $r$ at time $k$ are respectively denoted by $X_{i,k} \in \mathbb{R}$ and $X_{r,k} \in \mathbb{R}$, $i \in \{1, 2\}$ and $r \in \{u, v\}$. The received signal of relay $r$ at time $k$ is

$Y_{r,k} = h_{s_i,r}X_{1,k} + h_{s_2,r}X_{2,k} + Z_{r,k}, \quad r \in \{u, v\}, \quad k \in \mathbb{N}$,

where $(h_{s_i,r})$ is a continuous random variable, and $(Z_{r,k})$ is a zero-mean Gaussian random variable with variance $\sigma^2$.

The outerbound is established by the following bounds:

- **Bounding the end-to-end mutual information**: We analyze the three bounds and then proceed to prove three outerbounds that effectively capture the tension between these groups. Analyzing the three bounds yields that the sum-DoF is upper bounded by 4/3 almost surely.

The outerbound and the sum-DoF of a single-hop IC are given by

- **Sum-DoF of a single-hop IC**: The sum-DoF of a single-hop IC is $\frac{4}{3}$.

The sum-DoF of a single-hop IC is $\frac{4}{3}$.
and for destination $d_i$, the received signal at time $k$ is

$$Y_{i,k} = h_{u,d_i}X_{u,k} + h_{v,d_i}X_{v,k} + Z_{d_i,k}, \quad i \in \{1, 2\}, \quad k \in \mathbb{N},$$

where $Z_{u,k}$'s and $Z_{d_i,k}$'s are i.i.d (over time and with respect to each other) noise terms distributed as $\sim \mathcal{N}(0, 1)$, which are also independent of the messages $\{W_1, W_2\}$. We will use $X^n$ to denote a random column vector $[X_1 \ X_2 \ \ldots \ \ X_n]^T$. Also, for any $S \subseteq \{1, 2, \ldots, n\}$, we let $X^S$ denote $\{X_k | k \in S\}$.

**Definition 1.** An $(n, R_1, R_2)$-scheme with power constraint $P$ on the two-hop IC consists of the following:

1. A message set $W_i = \{1, 2, \ldots, 2^{R_i}\}$ at $s_i$ ($i \in \{1, 2\}$).
2. An encoding function $f_i: W_i \rightarrow X^n_i$ for each source $s_i$, $i \in \{1, 2\}$, such that $X^n_i = f_i(W_i)$, and every codeword $x^n_i$ satisfies the power constraint $\sum_{k=1}^{n} x^2_i(k) \leq nP$.
3. A relaying function $f_{r,k}: Y^n_{r,k} \rightarrow X_r$ at $r$ for each $r \in \{u, v\}$ and each $k \in \{1, 2, \ldots, n\}$, such that $X_{r,k} = f_{r,k}(Y^n_{r,k})$. In addition, every codeword $x^n_r$ should satisfy the power constraint $\sum_{k=1}^{n} x^2_r(k) \leq nP$.
4. A decoding function $g_{i}: Y^n_i \rightarrow W_i$ for destination $d_i$, $i \in \{1, 2\}$, such that $W_i = g_i(Y^n_i)$.
5. The error probability $P_e^n$ of the scheme is defined as $P_e^n = \Pr \left( \bigcup_{i=1}^{2} \{ W_i \neq \hat{W}_i \} \right)$, where each $W_i$ is chosen independently and uniformly at random from $\{1, 2, \ldots, 2^{R_i}\}$, $i \in \{1, 2\}$.

**Definition 2.** (Time-varying AF scheme) An $(n, R_1, R_2)$-scheme on the two-hop IC is called a time-varying AF if there exist $\{\mu_k \in \mathbb{R}\}_{k=1}^{n}$ and $\{\lambda_k \in \mathbb{R}\}_{k=1}^{n}$ such that, for each $k \in \{1, 2, \ldots, n\}$, $f_{u,k}(Y^n_{u,k}) = \mu_kY_{u,k-1}$ and $f_{v,k}(Y^n_{v,k}) = \lambda_k Y_{v,k-1}$.

**Definition 3.** A rate pair $(R_1, R_2)$ is time-varying-AF-achievable on $(U, V)$ if there exists a sequence of $(n, R_1, R_2)$-schemes that are time-varying AF on $(U, V)$, s.t. $\lim_{n \rightarrow \infty} P_e^n = 0$.

**Definition 4.** The sum-DoF achievable by time-varying AF, denoted by $D$, is defined by

$$D = \sup_{R_1, R_2} \lim_{P \rightarrow \infty} \min \left\{ \frac{R_1 + R_2}{2 \log_2 P} \mid (R_1, R_2) \text{ is time-varying-AF-achievable on } (U, V) \right\}.$$ 

The main result of the paper is the following theorem.

**Theorem 1.** The sum-DoF of two-hop IC with time-varying AF schemes is 4/3 for almost all values of channel gains.

In particular, the channel gain conditions needed for Theorem 1 to yield 4/3 sum-DoF are as follows:

1. All channel gains are non-zero.
2. $\text{rank}(H_1) = 2, \quad i \in \{1, 2\}$.
3. $\text{rank} \left( \begin{bmatrix} h_{u,d_i} & h_{v,d_i} \\ h_{u,d_i} & h_{v,d_i} \end{bmatrix} \right) = 2, \quad i \in \{1, 2\}, \quad i = 3 - i.$ (1)

It is easy to see that almost all values of channel gains satisfy the above conditions. In the rest of the paper, in which we prove Theorem 1, we assume that conditions (c-1)-(c-3) hold.

**III. ACHIEVING 4/3 SUM-DOF BY TIME-VARYING AF**

The achievability scheme consists of three phases, during which each source sends two distinct symbols, and at the end of the three phases each receiver is able to reconstruct an interference-free, but noisy, version of its desired symbols.

First note that, for time-varying AF strategies, the received signals at the destinations at each time $k$ can be written as

$$\begin{align*}
Y_{1,k} &= H_2 \begin{bmatrix} \mu_k & 0 \\ 0 & \lambda_k \end{bmatrix} H_1 \begin{bmatrix} X_{1,k-1} \\ X_{2,k-1} \end{bmatrix} + Z_{1,k} \\
Y_{2,k} &= H_2 \begin{bmatrix} \mu_k & 0 \\ 0 & \lambda_k \end{bmatrix} H_1 \begin{bmatrix} X_{1,k-1} \\ X_{2,k-1} \end{bmatrix} + Z_{2,k} 
\end{align*}$$

where $\mu_k$ and $\lambda_k$ are the AF coefficients at time $k$, $Z_{i,k} = h_{u,d_i} \mu_k Z_{u,k-1} + h_{v,d_i} \lambda_k Z_{v,k-1} + Z_{d_i,k}$ is the effective noise at destination $d_i, i \in \{1, 2\}$, and $G_k = H_2 \begin{bmatrix} \mu_k & 0 \\ 0 & \lambda_k \end{bmatrix} H_1$ is the equivalent end-to-end channel matrix given by

$$G_k = \begin{bmatrix} \mu_k h_{u,d_1} h_{s_1,u}, \ldots, \mu_k h_{u,d_1} h_{s_1,u}, & \mu_k h_{u,d_2} h_{s_1,u} + \lambda_k h_{v,d_1} h_{s_1,v}, & \mu_k h_{u,d_1} h_{s_2,u} + \lambda_k h_{v,d_1} h_{s_2,v} \\
\mu_k h_{u,d_2} h_{s_1,u}, \ldots, \mu_k h_{u,d_2} h_{s_1,u}, & \mu_k h_{u,d_2} h_{s_1,u} + \lambda_k h_{v,d_2} h_{s_1,v}, & \mu_k h_{u,d_2} h_{s_2,u} + \lambda_k h_{v,d_2} h_{s_2,v} \end{bmatrix}.$$  

For notational convenience, let $G_k = \begin{bmatrix} \alpha_{1,k} & \beta_{1,k} \\ \alpha_{2,k} & \beta_{2,k} \end{bmatrix}$. Then, the received signal at destination $d_i, i \in \{1, 2\}$, at time $k$ is

$$Y_{i,k} = \alpha_{i,k} X_{1,k} + \beta_{i,k} X_{2,k} + Z_{i,k}, \quad k \in \{1, 2, \ldots, n\}. \quad (4)$$

Note that the variance of $Z_{i,k}$ depends only on channel coefficients and amplifying factors (chosen from $(U, V)$), therefore it does not scale with $P$.

We will now describe the three phases of our time-varying AF achievability scheme in details. Set $U = \{c\}$, and $V = \{0, -c h_{u,d_1} h_{s_2,u}/h_{u,d_1} h_{s_2,v}, -c h_{u,d_2} h_{s_2,u}/h_{v,d_2} h_{s_2,v}\}$, where the constant $c \in \mathbb{R}$ is chosen to satisfy the power constraint $P$ at the relays. More specifically,

$$c = \min \left\{ \sqrt{1/(h_{s_1,u}^2 + h_{s_2,u}^2 + 1)}, \sqrt{1/(h_{s_1,v}^2 + h_{s_2,v}^2 + 1)} \right\},$$

where

$$l = \min\{ h_{u,d_1} h_{s_2,v}/h_{u,d_1} h_{s_1,u}, h_{v,d_1} h_{s_2,v}/h_{v,d_1} h_{s_2,v} \}.$$ 

Note that the denominators are non-zero by condition (c-1).

**Phase 1.** In this phase, $s_1$ and $s_2$ send two symbols $a_1$ and $b_1$, respectively $(a_1^2, b_1^2 \leq P)$. We choose the AF factors at the relays such that the interference from $s_2$ is canceled at $d_1$. More specifically, we set $\mu_1 = c$ and $\lambda_1 = -c h_{u,d_1} h_{s_2,u}/h_{u,d_1} h_{s_2,v}$. By inserting this choice of $\lambda_1$ and $\mu_1$ in (4), $d_1$ and $d_2$ will respectively receive

$$y_{1,1} = \alpha_{1,1} a_1 + z_{1,1}, \quad y_{2,1} = \alpha_{2,1} a_1 + \beta_{2,1} b_1 + z_{2,1}, \quad (5)$$

where $\alpha_{1,1} \neq 0$ and $\beta_{2,1} \neq 0$ (due to conditions (c-1), (c-2), and (c-3) in (1)), and $L_1(a_1, b_1)$ indicates a linear equation in $a_1$ and $b_1$. Thus, as shown in Figure 2(a), $d_1$ and $d_2$ now respectively have noisy versions of $a_1$ and $L_1(a_1, b_1)$.

**Phase 2.** In this phase, $s_1$ and $s_2$ send two new symbols $a_2$ and $b_2$ $(a_2^2, b_2^2 \leq P)$. However, this time, we cancel the effect of $s_1$...
at $d_2$, by letting $\mu_2 = c$ and $\lambda_2 = -c h_{u,d_2} h_{s_1,u}/h_{v,d_2} h_{s_1,v}$. Then $d_1$ and $d_2$ will respectively receive
\begin{align*}
y_{1,2} &= \alpha_{1,2} a_2 + \beta_{1,2} b_2 + z_{1,2}, \quad \text{and} \quad y_{2,1} = \beta_{2,2} b_2 + z_{2,2}, \tag{6}
\end{align*}
where $\alpha_{1,2} \neq 0$ and $\beta_{2,2} \neq 0$ (due to conditions (c-1), (c-2), and (c-3) in (1)), and $L_2(a_2, b_2)$ indicates a linear equation in $a_2$ and $b_2$. Thus, as shown in Figure 2(b), $d_1$ and $d_2$ now respectively have noisy versions of $L_2(a_2, b_2)$ and $b_2$.

**Phase 3.** Now notice that, if, at phase 3, destination $d_3$ receives a linear combination of $a_3$ and $b_2$ ($L_3(a_3, b_2)$), then it can solve for (a noisy version of) $a_3$ given equations (5) and (6). Similarly, if $d_2$ receives $L_3(a_1, b_2)$, then it can also solve for (a noisy version of) $b_1$ given equations (5) and (6). Thus, as shown in Figure 2(c), in phase 3, $s_1$ sends $a_1$, $s_2$ sends $b_2$, and we choose $\mu_3 = c$, and $\lambda_3 = 0$, so that $d_1$ and $d_2$ receive
\begin{align*}
y_{1,3} &= \alpha_{1,3} a_1 + \beta_{1,3} b_2 + z_{1,3}, \tag{7}
\end{align*}
and
\begin{align*}
y_{2,3} &= \alpha_{2,3} a_1 + \beta_{2,3} b_2 + z_{2,3}, \tag{8}
\end{align*}
where $\beta_{1,3} \neq 0$, and $\alpha_{2,3} \neq 0$ (due to condition (c-1) in (1)). Therefore, after the three phases, $d_1$ can construct
\begin{align*}
y_{a_1} &= a_1 + z_{1,1}/\alpha_{1,1}, \tag{9}
\end{align*}
and
\begin{align*}
y_{a_2} &= a_2 + \frac{1}{\alpha_{1,2}} z_{1,2} - \frac{\beta_{1,2}}{\alpha_{1,2} \beta_{1,3}} z_{1,3} + \frac{\alpha_{1,3} \beta_{1,2}}{\alpha_{1,2} \alpha_{1,2} \beta_{1,3}} z_{1,1}, \tag{10}
\end{align*}
from $(y_{1,1,1,2,1,3})$. Let $\sigma_1^2$ and $\sigma_2^2$ be the variances of the noise terms in equations (9) and (10). Note that they depend only on channel coefficients and AF factors. Hence, they are constants that do not scale with $P$. Then, by using a proper outercode, we can achieve a rate of
\begin{align*}
R_1 = \frac{1}{6} \left( \log \left( 1 + \frac{P}{\sigma_1^2} \right) + \log \left( 1 + \frac{P}{\sigma_2^2} \right) \right) = \frac{1}{3} \log \frac{P}{\sigma_1 \sigma_2}.
\end{align*}
So $d_1$ can achieve 2/3 DoF. Similarly, $d_2$ can also achieve 2/3 DoF, hence achieving a total of 4/3 sum-DoF. Note that a similar achievability scheme was used in the context of delayed CSIT analysis in [6].

**IV. OUTERBOUNDS ON DOF OF TIME-VARYING AF**

Consider a time-varying AF $(n, R_1, R_2)$-scheme $C$ with power constraint $P$, and error probability $P_\epsilon^n$ such that $P_\epsilon^n \to 0$ as $n \to \infty$. We will prove that $R_1 + R_2 \leq (2/3) \log P + o(\log P)$. Let $\mu_k$ and $\lambda_k$ denote the amplifying factors of $C$ at time $k$ of relays $u$ and $v$, respectively. Consider the end-to-end channel matrix $G_k$ (defined in (3)) created by scheme $C$ at time $k$. Note that the $i$-th column (row) of $G_k$ ($i \in [1, 2]$) corresponds to a linear combination of columns of $H_1$ ($H_2$) with coefficients $\mu_k h_{u,d_i}$ and $\lambda_k h_{s_i,v}$ ($\mu_k h_{u,d_i}$ and $\lambda_k h_{v,d_i}$). Also, the entries of the main diagonal are linear combinations of the columns of $H^1$ (defined in (1)) with coefficients $\mu_k$ and $\lambda_k$; similarly, the entries of the counterdiagonal are linear combinations of the columns of $H^2$ (defined in (1)) with coefficients $\mu_k$ and $\lambda_k$. Since by conditions (c-1), (c-2), and (c-3), specified in (1), all channel coefficients are non-zero, and $H_1$, $H_2$, $H^1$, and $H^2$ have full rank, then no pair of entries in $G_k$ can be zero, unless $\lambda_k = \mu_k = 0$. Therefore, at each time $k$ either $G_k$ has at most one zero entry or $G_k = 0$. As a result, if $G_k$ is non-zero at any time $k$, then it belongs to one of the states shown in Figure 3. Asterisks denote non-zero entries. We denote the collective state $(C_1, C_2, C_3)$ by $C$.

Define the scaled received signals:
\begin{align*}
\tilde{Y}_{i,k} &= \frac{1}{\mu_k + |\lambda_k| + 1} Y_{i,k}. \tag{11}
\end{align*}
Note that since the code is fixed, then $\mu_k$’s and $\lambda_k$’s are fixed for all $k$. So the above operation is well defined, and it is clear that for any set $L \subset \{1, \ldots, n\}$, $I(X^L; Y^L) \leq I(X^L, \tilde{Y}^L)$. Also, define
\begin{align*}
\tilde{\mu}_k &= \frac{\mu_k}{|\mu_k| + |\lambda_k| + 1}, \quad \text{and} \quad \tilde{\lambda}_k &= \frac{\lambda_k}{|\mu_k| + |\lambda_k| + 1}. \tag{12}
\end{align*}
Note that $\tilde{\mu}_k, \tilde{\lambda}_k \in [-1, 1]$. Let $I = [-1, 1]$. So we can write
\begin{align*}
\tilde{Y}_{1,k} &= G_{k} \begin{bmatrix} X_{1,k-1} \\ Y_{2,k} \end{bmatrix}, \quad \text{and} \quad \tilde{Z}_{1,k} &= G_{k} \begin{bmatrix} Z_{1,k-1} \\ Z_{2,k} \end{bmatrix}, \tag{13}
\end{align*}
where $\tilde{G}_k$ is given similarly to equation (3), by replacing $\mu_k$ and $\lambda_k$ by $\tilde{\mu}_k$ and $\tilde{\lambda}_k$, respectively. Also, $\tilde{Z}_{1,k} = h_{u,d_i} \tilde{\mu}_k Z_{u,d_i} Z_{u,k-1}$ and $\tilde{Z}_{2,k} = h_{v,d_i} \tilde{\lambda}_k Z_{v,d_i} Z_{v,k-1} + \tilde{Z}_{2,k}$. Again, for notational convenience, write $\tilde{G}_k = \begin{bmatrix} \tilde{G}_{1,k} & \tilde{G}_{2,k} \end{bmatrix} \begin{bmatrix} \tilde{\lambda}_{1,k} \tilde{\mu}_{1,k} \\ \tilde{\lambda}_{2,k} \tilde{\mu}_{2,k} \end{bmatrix}$. 

![Diagram](image-url)
Similarly to equation (4), we will write the vector of $n$ received signals at destination $d_i$ (with an abuse of notation)
\[
\tilde{Y}_i^n = \hat{\alpha}_i^n X_i^n + \hat{\beta}_i^n X_{2i}^n + \hat{Z}_i^n, \quad i \in \{1, 2, \ldots, n\},
\]
where $\hat{\alpha}_i^n$ and $\hat{\beta}_i^n$ are understood as $n \times n$ diagonal matrices, where the $j^{th}$ entries of the diagonals are respectively $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ ($i \in \{1, 2, \ldots, n\}, j \in \{1, \ldots, n\}$). Similarly, for any $L \subset \{1, 2, \ldots, n\}$, we write
\[
\tilde{X}_i^L = \tilde{\alpha}_i^L X_i^L + \tilde{\beta}_i^L X_{2i}^L + \tilde{Z}_i^L,
\]
where $\tilde{\alpha}_i^L$ and $\tilde{\beta}_i^L$ are $|L| \times |L|$ diagonal matrix, whose diagonal entries are respectively $\{\tilde{\alpha}_{i,j}\}_{i \in L}$ and $\{\tilde{\beta}_{i,j}\}_{i \in L}$ ($i \in \{1, 2, \ldots, n\}$).

Now, for any code $C$ (with AF coefficients $\lambda_k$ and $\mu_k$, $k = 1, \ldots, n$), we define the set $A_C$ as
\[
A_C = \{k \in \{1, 2, \ldots, n\} : \mu_k h_{u_d,k} + \lambda_k h_{v_d,k} = 0\}.
\]
Similarly, let $B_C$, $C_C$, $C_{1,C}$, $C_{2,C}$, and $C_{3,C}$ be the sets of time indices corresponding to states $B$, $C$, $C_1$, $C_2$, and $C_3$, respectively. Also, let $S_C = A_C \cup B_C \cup C_C$. So that $S_C = \{k \in \{1, 2, \ldots, n\} : \mu_k = \lambda_k = 0\}$. Note that, since the channel gains are fixed, and we are considering a specific scheme $C$ which fixes $\mu_k$ and $\lambda_k$ for all $k$, then the previously defined sets are deterministic, and thus well-defined. As the ease of notation, we will drop the subscript $C$ in the rest of this section, and refer to those sets as $A$, $B$, $C$, $C_1$, $C_2$, $C_3$, and $S$.

We now state our main lemma which yields $D \leq 4/3$.

**Lemma 1.** For any time-varying AF $(n, R_1, R_2)$-scheme $C$, with power constraint $P$ and associated sets $A, B, C, C_1, C_2, C_3$, and $S$. We now state our main lemma which yields $D \leq 4/3$.

**Proof of Bound (1) in Lemma 1**
Recall that $S = A \cup B \cup C$. We claim that $I(X_i^n; \tilde{Y}_i^n) = I(X_i^S; \tilde{Y}_i^n)$, since:
\[
I(X_i^n; \tilde{Y}_i^n) = I(X_i^S; \tilde{Y}_i^n)
\]
where (a) follows from the fact that $\tilde{Y}_i^S$ is a function of messages and noise at other time steps, and (b) follows from the fact that $X_i^S$ is a function of $W_i$, hence independent of $W_2$ (and $X_3^S$) and of noise. Similarly, $I(X_2^n; \tilde{Y}_2^n) = I(X_2^S; \tilde{Y}_2^n)$. Now, using Fano and (17), we get
\[
n(R_1 + R_2) \leq I \left( X_1^n; \tilde{Y}_1^n \right) + I \left( X_2^n; \tilde{Y}_2^n \right) + n \epsilon_n
\]
\[
= I \left( X_1^n; \tilde{Y}_1^n \right) + I \left( X_2^n; \tilde{Y}_2^n \right) + n \epsilon_n
\]
where $\epsilon_n \to 0$, as $P_n \to 0$. Now, we bound the last two terms:
\[
I(X_1^n; \tilde{Y}_1^n) \leq h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n | Y_1^n, X_1^n)
\]
\[
\leq h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n)
\]
where $i \in \{1, 2\}$, $i = 3 - i$, (a) follows from the fact that noise is independent of $W_1, W_2$, and of noise terms at other time steps, and (b) follows since
\[
h(\tilde{Z}_1^n) \geq h(\tilde{Z}_1^n | Z_1^n, Z_2^n) = h(\tilde{Z}_1^n) \geq 0.
\]
Now, to bound the first two terms in (18), consider the following chain of inequalities.
\[
I(X_1^n; \tilde{Y}_1^n) = I(X_2^n; \tilde{Y}_2^n)
\]
\[
\leq h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n | X_1^n)
\]
\[
+ h(Y_2^n) - h(\tilde{Y}_2^n | X_2^n)
\]
\[
= h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n)
\]
where $\tau_1, \tau_2$, and $\tau_3$ are constants that do not depend on $P$.

Before proving Lemma 1, we first demonstrate how it yields $D \leq 4/3$. Suppose that the Lemma is true. Then, by taking the minimum of the three bounds, we get
\[
R_1 + R_2 \leq \min_{L \subseteq \{A, B, C\}} \frac{1}{2} \left( 1 + \frac{|L|}{n} \right) \log_2 P + \tau_1,
\]
\[
R_1 + R_2 \leq \min_{L \subseteq \{A, B, C\}} \frac{1}{2} \left( 1 + \frac{|L|}{n} \right) \log_2 P + \tau_2,
\]
\[
R_1 + R_2 \leq \min_{L \subseteq \{A, B, C\}} \frac{1}{2} \left( 1 + \frac{|L|}{n} \right) \log_2 P + \tau_3,
\]
where $\tau_1, \tau_2, \tau_3$ are constants that do not depend on $P$.

We now state our main lemma which yields $D \leq 4/3$.

**Proof of Bound (1) in Lemma 1**
Recall that $S = A \cup B \cup C$. We claim that $I(X_i^n; \tilde{Y}_i^n) = I(X_i^S; \tilde{Y}_i^n)$, since:
\[
I(X_i^n; \tilde{Y}_i^n) = I(X_i^S; \tilde{Y}_i^n)
\]
where (a) follows from the fact that $\tilde{Y}_i^S$ is a function of messages and noise at other time steps, and (b) follows from the fact that $X_i^S$ is a function of $W_i$, hence independent of $W_2$ (and $X_3^S$) and of noise. Similarly, $I(X_2^n; \tilde{Y}_2^n) = I(X_2^S; \tilde{Y}_2^n)$. Now, using Fano and (17), we get
\[
n(R_1 + R_2) \leq I \left( X_1^n; \tilde{Y}_1^n \right) + I \left( X_2^n; \tilde{Y}_2^n \right) + n \epsilon_n
\]
\[
= I \left( X_1^n; \tilde{Y}_1^n \right) + I \left( X_2^n; \tilde{Y}_2^n \right) + n \epsilon_n
\]
where $\epsilon_n \to 0$, as $P_n \to 0$. Now, we bound the last two terms:
\[
I(X_1^n; \tilde{Y}_1^n) \leq h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n | Y_1^n, X_1^n)
\]
\[
\leq h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n)
\]
where $i \in \{1, 2\}$, $i = 3 - i$, (a) follows from the fact that noise is independent of $W_1, W_2$, and of noise terms at other time steps, and (b) follows since
\[
h(\tilde{Z}_1^n) \geq h(\tilde{Z}_1^n | Z_1^n, Z_2^n) = h(\tilde{Z}_1^n) \geq 0.
\]
Now, to bound the first two terms in (18), consider the following chain of inequalities.
\[
I(X_1^n; \tilde{Y}_1^n) = I(X_2^n; \tilde{Y}_2^n)
\]
\[
\leq h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n | X_1^n)
\]
\[
+ h(Y_2^n) - h(\tilde{Y}_2^n | X_2^n)
\]
\[
= h(\tilde{Y}_1^n) - h(\tilde{Y}_1^n)
\]
where $\tau_1, \tau_2, \tau_3$ are constants that do not depend on $P$.

We now state our main lemma which yields $D \leq 4/3$.

**Lemma 2.** Let $X, Y, Z$ be two random vectors of size $n$, such
that $X \independent (Y, Z)$. Let $M$ and $M'$ be two $n \times n$ constant invertible matrices. Then

\[ h(MX + Y) - h(M'X + Z) \leq h(M'M^{-1}Y - Z) - h(Z|Y) - \log |\det (M'M^{-1})|. \]

Proof:

\[ h(MX + Y) - h(M'X + Z) = h(M'X + M'M^{-1}Y) - h(M'X + Z) - \log |\det (M'M^{-1})| \]

\[ \leq h(M'X + M'M^{-1}Y) - h(M'X + Z|M'M^{-1}Y - Z) - \log |\det (M'M^{-1})| \]

\[ = -h(M'X + M'M^{-1}Y|M'M^{-1}Y - Z) + h(M'M^{-1}Y) - \log |\det (M'M^{-1})| \]

\[ = I(M'X + M'M^{-1}Y; M'M^{-1}Y - Z) - \log |\det (M'M^{-1})| \]

\[ = (M'M^{-1}Y - Z) - h(M'M^{-1}Y - Z|X, Y) - \log |\det (M'M^{-1})| \]

\[ \leq h(M'M^{-1}Y - Z) - h(M'M^{-1}Y - Z|X, Y) - \log |\det (M'M^{-1})| \]

\[ \leq h(M'M^{-1}Y - Z) - h(Z|Y) - \log |\det (M'M^{-1})|. \]

Then we can apply Lemma 2 on the bracketed terms in equation (21), where for the first term \( \{X = X^B, Y = Z^B, Z = Z^B, M = \tilde{Z}^B, \tilde{M} = \tilde{Z}^B, \text{and for the second term} \{X = X^A, Y = Z^A, Z = Z^A, M = \tilde{Z}^A, \tilde{M} = \tilde{Z}^A\}. \) So by setting \( M_1 = (\tilde{Z}^A)(\tilde{Z}^A)^{-1}, M_2 = \tilde{Z}^A(\tilde{Z}^A)^{-1}, \) we get

\[ I(X_1; X_1, \tilde{X}_2, Y_2) + I(X_2; X_2, \tilde{X}_2, Y_2) \leq h(\tilde{Y}_1) + h(\tilde{Y}_2) - h(\tilde{Z}_1^B) - h(\tilde{Z}_2^B) \]

\[ + h\left(M_1^1 \tilde{Z}_1^B - \tilde{Z}_2^B\right) - \log |\det (M_1)| \]

\[ + h\left(M_2^2 \tilde{Z}_2^A - \tilde{Z}_1^A\right) - \log |\det (M_2)|. \]

\[ (22) \]

Recall that \( \tilde{Z}^B \) is an \( |B| \times |B| \) diagonal matrix whose diagonal entries are \( \{\tilde{a}_k\}_{k \in B}. \) The same holds for \( \tilde{Z}^A, \tilde{B}_1, \) and \( \tilde{B}_2. \) Therefore, we get that \( M \) is a diagonal matrix whose diagonal entries are \( \{\tilde{a}_{k, k}\}_{k \in B}. \) Now note that, for any \( k \in B, \) we have \( \tilde{\mu}_k h_{d_1, d_2} h_{s_2, u} + \tilde{\lambda}_k h_{v_1, d_2} h_{s_2, v} = 0, \) i.e. \( \tilde{\mu}_k = -\tilde{\lambda}_k (h_{d_1, d_2} h_{s_2, u})/(h_{u_1, d_2} h_{s_2, u}); \) which yields that, for any \( k \in B, \) the ratio \( \tilde{\alpha}_{k, k}/\tilde{\alpha}_{l, l} \) is equal to a constant \( c, \) which is independent of \( \tilde{\mu}_k \) and \( \tilde{\lambda}_k. \) (since they simplify in the expression of the ratio, check (3)).

The same holds for \( M_2; \) therefore we can bound the summation of the log terms in (22) by \( \gamma_{1, 1} n, \) where \( \gamma_{1, 1} \) is independent of \( n \) and \( P. \)

Also it easy to see that \( h(M_1^1 \tilde{Z}_1^B - \tilde{Z}_2^B) \) and \( h(\tilde{Z}_1^A|\tilde{Z}_2^A) \) are positive. Also, consider the following:

\[ h(M_1^1 \tilde{Z}_1^B - \tilde{Z}_2^B) \]

\[ \leq \sum_{k \in B} h\left(\tilde{Z}_{1,k} - \tilde{Z}_{2,k}\right) \]

\[ = \sum_{k \in B} \left( h\left((ch_{u,d} - h_{u,d}) \tilde{\mu}_k h_{s_2, u} + (c h_{v,d} - h_{v,d}) \tilde{\lambda}_k h_{s_2, v}\right) \right) \]

\[ \leq \sum_{k \in B} \frac{1}{2} \log_2 \left(2\pi e \left( (ch_{u,d} - h_{u,d})^2 \tilde{\mu}_k^2 + (c h_{v,d} - h_{v,d})^2 \tilde{\lambda}_k^2 \right) + \frac{1 + c^2}{|\tilde{\mu}_k| + |\tilde{\lambda}_k| + 1} \right) \]

\[ \leq \frac{\gamma_{1, 2} n}{2} \]

(23)

Now, we bound \( h(\tilde{Y}_1) \) by

\[ h(\tilde{Y}_1) - |A| \log_2 (2\pi e)/2 \leq \sum_{k \in A} h(\tilde{Y}_{1,k}) - |A| \log_2 (2\pi e)/2 \]

\[ \leq \sum_{k \in A} \frac{1}{2} \log_2 \left(2\pi e \left( (ch_{u,d} - h_{u,d})^2 \tilde{\mu}_k^2 + (c h_{v,d} - h_{v,d})^2 \tilde{\lambda}_k^2 \right) + \frac{1 + c^2}{|\tilde{\mu}_k| + |\tilde{\lambda}_k| + 1} \right) \]

\[ \leq \frac{\gamma_{1, 2} n}{2} \]

(24)

where (a) follows from the fact that variance of the sum of pairwise independent random variables is the sum of the individual variances, (b) follows from the fact that \( \tilde{\mu}_k, \tilde{\lambda}_k \in [-1, 1], \) and \( \gamma_{1, 2} \) independent of \( n \) and \( P. \)

Now, by equations (18), (19), (22), and (23), we get

\[ n(R_1 + R_2) \leq h(\tilde{Y}_1) + h(\tilde{Y}_2) + h(\tilde{Y}_3) + h(\tilde{Y}_4) + \gamma_{1, 2} n, \]

(25)

where \( \gamma_{2} \) is independent of \( n \) and \( P. \)
where \( (a) \) follows from Jensen’s inequality, and \( (b) \) follows from the power constraint \( P \). Now, notice that
\[
(N + 2Mn/|A|)^{|A|} = N^{|A|} \left(1 + \frac{2Mn}{N} \right)^{|A|} \leq N^{|A|}(1 + 2Mn/N)^n,
\]
where \( (a) \) follows from the fact that the sequence \((1 + x/m)^m\) is monotonically increasing in \( m \), when \( x > 0 \). Then
\[
|A|\log_2(N + 2Mn/|A|) \leq n\log_2(1 + 2M/N) + |A|\log_2 N,
\]
which, combined with equation (27), yields
\[
h(Y^A) \leq \frac{|A|}{2} \log_2 P + \gamma_3 n,
\]
where \( \gamma_3 \) is a constant that does not depend on \( P \). Similarly
\[
h(Y^B) \leq \frac{|B|}{2} \log_2 P + \gamma_4 n,
\]
\[
h(Y^C_i) \leq \frac{|C_i|}{2} \log_2 P + \gamma_5 n, \quad i \in \{1, 2\},
\]
where \( \gamma_4, \gamma_5,1 \), and \( \gamma_5,2 \) are constants that do not depend on \( P \). So, from equations (24), (28), (29), and (30) we get
\[
n(R_1 + R_2) \leq \frac{1}{n}(|S| + |C|)\log_2 P + \tau_1 n
\]
\[
\leq \frac{n}{2} \left(1 + \frac{|C|}{n}\right) \log_2 P + \tau_1 n,
\]
where \( \tau_1 \) is a constant that does not depend on \( P \).

**Proof of Bound (2) in Lemma 1**

Define the set \( E = C_1 \cup C_2, \) and consider the following.
\[
n(R_1 + R_2) - n_{e_0} \leq (a) I(X^n_1; \tilde{Y}_n^1) + I(X^n_2; \tilde{Y}_n^2)
\]
\[
(b) I(X^n_1; \tilde{Y}_n^1) + I(X^n_2; \tilde{Y}_n^2) \leq I(X^n_1; \tilde{Y}_n^1) + I(X^n_2; \tilde{Y}_n^2) + I(X^n_1; \tilde{Y}_n^1|X^n_2)
\]
\[
\leq h(Y^S_1) - h(\tilde{Y}^A_1 X^A_1 + Z^A_1, \tilde{Y}^B_1 X^B_1 + Z^B_1) - h(Z^C_2)
\]
\[
+ h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2)
\]
\[
\leq \left[h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2)ight]
\]
\[
(h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2)) (T1)
\]
\[
-h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2) - h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(Z^C_2)
\]
\[
(T2)
\]
\[
 where \( (a) \) follows from Fano’s inequality, \( (b) \) follows from equation (17), and \( (c) \) follows from the independence of \( W_1 \) and \( W_2 \). Now, we will bound the term \( (T1) \). First, set \( M_2 = (\tilde{\beta}^A_1)(\tilde{\beta}^A_1)^{-1} \), and \( M_3 = (\tilde{\beta}^E_1)(\tilde{\beta}^E_1)^{-1} \). Then note
\[
h(\tilde{Y}^B_1) + h(\tilde{Y}^C_1) - h(\til{Y}^B_1) + h(\til{Y}^C_1) - h(Z^C_2) - h(\til{Y}^B_1) + h(\til{Y}^C_1) - h(Z^C_2)
\]
\[
\leq \log \left| \det(M_2) \right| \det(M_3)
\]
\[
(T3)
\]
\[
where the inequality follows from Lemma 2 and the fact that \( h(MX, Y) = h(X, Y) + \log |\det M| \). Now, we bound \( T2 \):
\[
h(\til{Y}^B_1) + h(\til{Y}^C_1) - h(\til{Y}^B_1) + h(\til{Y}^C_1) - h(Z^C_2) - h(\til{Y}^B_1) + h(\til{Y}^C_1) - h(Z^C_2)
\]
\[
\leq \log \left| \det(M_2) \right| \det(M_3)
\]
\[
(33)
\]
Then, by equations (31), (32), and (33), we get
\[
n(R_1 + R_2) \leq h(Y^S_1) + h(Y^B_1) - h(Z^C_2) - h(M_2 \til{Y}^A_1, M_3 \til{Z}^E_1 - \til{Z}^F_1)
\]
\[
(34)
\]
where \( \gamma_6 \) is a constant that does not depend on \( P \). Now, similarly to (28), we bound \( h(Y^S_1) \) as
\[
h(Y^S_1) \leq \frac{|S|}{2} \log_2 P + \gamma_7 n
\]
\[
(35)
\]
where \( \gamma_7 \) is a constant that does not depend on \( P \). Then, from equations (34), (35), and (29), we get
\[
n(R_1 + R_2) \leq \frac{1}{n}(|S| + |B|) \log_2 P + \tau_2 n
\]
\[
\leq \tau_2
\]
where \( \tau_2 \) is a constant that does not depend on \( P \).

The proof of the third bound is similar, and thus omitted.

V. CONCLUSION

In this paper, we analyzed the sum-DoF of the two-hop IC with constant coefficients, where relays are restricted to perform time-varying AF schemes. We showed that 4/3 sum-DoF is achievable using such schemes, as opposed to constant AF schemes that achieve at most 1. We also proved a matching outerbound, hence characterizing the sum-DoF of this network with time-varying AF strategies. This study can be extended in several directions. One direction could be to consider the impact of time-varying AF strategies in more general two-unicast networks, such as the layered two-unicast networks studied in [7], or multi-flow networks studied in [5].

REFERENCES


