Transmit Beamforming in Rayleigh Product MIMO Channels: Ergodic Mutual Information and Symbol Error Rate

Zhong Zheng[†], Lu Wei[∗‡], Zygmunt J. Haas[†♯], and Vahid Tarokh[∗]
[†] Department of Computer Science, University of Texas at Dallas, U.S.A
Email: {zheng.zhong1, haas}@utdallas.edu
[∗] School of Engineering and Applied Sciences, Harvard University, U.S.A
Email: lwei@g.harvard.edu, vahid@seas.harvard.edu
[‡] Department of Mathematics and Statistics, University of Helsinki, Finland
[♯] School of Electrical and Computer Engineering, Cornell University, U.S.A

***Abstract*—In this paper, we consider MIMO beamforming in the presence of Rayleigh product channels. Based on a derived largest eigenvalue distribution, the key performance metrics of the beamforming system are obtained, assuming perfect channel knowledge at the transmitter and receiver. Using the closed-form expressions, we gain insights into the behavior of MIMO beamforming systems in scenarios of practical interest.**

I. Introduction

Transmit beamforming in MIMO communications exploits the channel knowledge at both transmitter and receiver. With Maximum Ratio Combining (MRC) at the receiver, the received Signal to Noise Ratio (SNR) is maximized by transmitting in the direction of eigenvector corresponding to the largest eigenvalue of the channel ([1]). Prior work on this topic has focused on the Rayleigh fading channels ([1]–[3]), where a rich scattering environment exists. However, in certain communication scenarios, the signal propagation may be subject to insufficient scattering ([4]) or keyhole effect ([5]), which exhibits rank deficiency. A MIMO model that captures these effects is the Rayleigh product channel, which is characterized by a matrix product of two statistically independent Rayleigh MIMO channels[1].

There exists a number of studies on the performance of MIMO beamforming over Rayleigh product channels. In [6], [7], the authors obtained the outage probability of MIMO beamforming. Therein, the ergodic capacity and average Symbol Error Rate (SER) are given as integrals over the largest eigenvalue distribution of the channel matrix. Assuming single transmit/receive antenna or a single keyhole MIMO channel, the explicit expressions of these performance metrics are derived. In the high SNR regime, an approximate average SER was given in [8] for arbitrary channel dimensions. When spatial correlations exist at the transmitter and the receiver, a low SNR capacity approximation was considered in [9] using statistical Channel State Information (CSI). The optimality condition to achieve ergodic capacity using MIMO beamforming was given in [10]. The authors in [11] studied the outage probability of a similar multi-keyhole MIMO channel with different keyhole amplitudes. The obtained results are only valid in the high/low SNR regime ([6], [8], [9]) or restricted to certain channel configurations ([6], [7]).

Using a similar technique as in [2], we propose a simple yet accurate approximation to the largest eigenvalue distribution of Rayleigh product channels, which is the perturbed version of the original distribution function. Based on this result, closed-form expressions of ergodic MI and average SER are calculated, assuming perfect CSI at the transmitter and the receiver. The proposed analytical framework is useful to investigate the performance of MIMO beamforming over Rayleigh product channels with practical channel configurations.

II. System Model

Consider a discrete time, baseband MIMO system with n_0 transmit and n_2 receive antennas. The transmitted symbols are assumed to experience a Rayleigh product channel with n_1 scattering objects. The end-to-end equivalent MIMO channel $\mathbf{H}$ is given by

$$\mathbf{H} = \frac{1}{\sqrt{n_1}}\mathbf{H}_2\mathbf{H}_1, \tag{1}$$

where $\mathbf{H}_1 \in \mathbb{C}^{n_1\times n_0}$ denotes the channels between the scattering objects and the transmit array, and $\mathbf{H}_2 \in \mathbb{C}^{n_2\times n_1}$ denotes the channels between the scattering objects and the receive array. The entries of $\mathbf{H}_1$ and $\mathbf{H}_2$ are independent complex Gaussian distributed with zero-mean and unit variance. In line with [4], [6], [8]–[10], the channel $\mathbf{H}$ is normalized by the constant $\sqrt{n_1}$, so that the total energy of the channel is equal to an AWGN channel with an array gain $\mathbb{E}[\mathrm{Tr}(\mathbf{H}^\dagger\mathbf{H})] = n_0 n_2$. We will hereafter parameterize the Rayleigh product channel (1) by the three-tuple (n_0, n_1, n_2).

The channel output $\mathbf{y}$, at a given time instance, equals

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{2}$$

where the additive noise $\mathbf{n}$ is modeled as an *i.i.d.* complex Gaussian vector with zero mean and variance σ_n^2, i.e., $\mathbf{n} \sim$

[1]A more general MIMO channel, i.e., the double-scattering channel, is considered in [4]. This channel model is characterized by a matrix product involving three deterministic matrices, i.e., transmit, receive, and scatterer correlation matrices, and two independent complex Gaussian matrices. As a special case, Rayleigh product model corresponds to the scenario where the antenna elements, as well as the scattering objects, are sufficiently separated such that the effective spatial correlations can be ignored.

$\mathcal{CN}(\mathbf{0}, \sigma_n^2 \mathbf{I})$. For a MIMO beamforming system, the transmit signal vector $\mathbf{x} = \mathbf{w}_{\mathrm{BF}}\, x$, where x is the information symbol with transmission power $\mathbb{E}[|x|^2] = P$. Denoting the matrix $\mathbf{P} = \mathbf{H}_2\mathbf{H}_1$, the beamforming vector $\mathbf{w}_{\mathrm{BF}}$ is chosen as the eigenvector corresponding to the largest eigenvalue[2] of $\mathbf{P}^\dagger\mathbf{P}$, which is also known as the power of the dominant eigenchannel of $\mathbf{H}$. At the receive array, the received signal $\mathbf{y}$ are linearly combined according to the MRC principle using the vector $\mathbf{w}_{\mathrm{BF}}^\dagger\mathbf{H}^\dagger$ such that

$$z = \mathbf{w}_{\mathrm{BF}}^\dagger\mathbf{H}^\dagger\mathbf{y} = \mathbf{w}_{\mathrm{BF}}^\dagger\mathbf{H}^\dagger\mathbf{H}\mathbf{w}_{\mathrm{BF}}x + \mathbf{w}_{\mathrm{BF}}^\dagger\mathbf{H}^\dagger\mathbf{n}. \tag{3}$$

The SNR after the MIMO beamforming (3) is given by

$$\mathrm{SNR} = \mathbf{w}_{\mathrm{BF}}^\dagger\mathbf{P}^\dagger\mathbf{P}\mathbf{w}_{\mathrm{BF}}\frac{P}{n_1\sigma_n^2} = \gamma\frac{\lambda_1}{n_1}, \tag{4}$$

where the λ_1 is the largest eigenvalue of $\mathbf{P}^\dagger\mathbf{P}$ and $\gamma = P/\sigma_n^2$ denotes the average SNR. Therefore, the performance of the MIMO beamforming system relies on the statistics of the largest eigenvalue λ_1. Note that in some asymptotic regime, the asymptotic distribution of λ_1 has been derived in [12] recently. Therein, the result can be considered a generalization of the Tracy-Widom law of the largest eigenvalue of the conventional Rayleigh MIMO channels. However, the asymptotic distribution function of λ_1 in [12] involves a second order differential equation, which may not be convenient for calculating the performance metrics of the MIMO beamforming system, such as the ergodic MI or average SER. In the next section, we obtain a simple approximation to the CDF of λ_1 for the finite dimensional channel matrices, which is amendable for performance analysis.

III. Largest Eigenvalue Distribution of Rayleigh Product Channels

A. Weak Commutation for Product of Random Matrices

It is convenient to define the three-tuple (r, s, t) as the permuted version of (n_0, n_1, n_2), such that $s \geq t \geq r$. Since the maximal rank of the matrix $\mathbf{P}$ is r, the Hermitian matrix $\mathbf{P}^\dagger\mathbf{P}$ has r non-zero eigenvalues such that $0 \leq \lambda_r \leq \cdots \leq \lambda_1 < \infty$. Denoting $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_r\}$, it was shown in [13, Eq. (43)] that the joint eigenvalue density function $f\left(\boldsymbol{\lambda}; (n_0, n_1, n_2)\right)$, parameterized by the matrix dimensions (n_0, n_1, n_2), is invariant under any permutation of n_0, n_1, and n_2. This property is referred to as a weak commutation relation for product of matrices in [13]. In particular, the non-zero eigenvalue density function of matrix $\mathbf{P}^\dagger\mathbf{P}$ is identical to an equivalent matrix specified by the three-tuple (r, s, t), i.e., $f\left(\boldsymbol{\lambda}; (n_0, n_1, n_2)\right) = f\left(\boldsymbol{\lambda}; (r, s, t)\right)$. In the following, we will work with the eigenvalue density function for the equivalent matrix (r, s, t), denoted as $f(\boldsymbol{\lambda})$, without specifying the dependence on the matrix dimensions.

According to [14, Eq. (18)], $f(\boldsymbol{\lambda})$ can be rewritten as[3]

$$f(\boldsymbol{\lambda}) = \frac{1}{c}\Delta(\lambda)\det\left(G_{0,2}^{2,0}\left(\begin{matrix} - \\ \nu_2,\, \nu_1 + j - 1\end{matrix}\,\middle|\,\lambda_i\right)\right), \tag{5}$$

where $\nu_1 = s - r$, $\nu_2 = t - r$, and c is a normalization constant. Here, we denote the Vandermonde determinant as $\Delta(\lambda) = \prod_{1\leq i<j\leq r}(\lambda_j - \lambda_i) = \det\left(\lambda_i^{j-1}\right)$. The Meijer's G-function is defined in (6) on top of the next page, where the contour $\mathcal{L}$ is chosen in such a way that the poles of $\Gamma(b_j + z)$, $j = 1, \ldots, m$ are separated from the poles of $\Gamma(1 - a_j - z)$, $j = 1, \ldots, n$. Note that in the special case $n_0 = n_1 = n_2$, the corresponding joint density $f(\boldsymbol{\lambda})$ has been derived in [15].

B. Perturbed Joint Eigenvalue Density

Distribution of the largest eigenvalue λ_1 can be readily calculated using the Andréief integral in [16] over the joint density function (5). However, the obtained distribution function, similar to the one in [6, Eq. (6)], is given in term of a determinant with matrix entries being special functions. Based on such formulation, it is difficult to obtain explicit expressions for some key performance metrics of MIMO beamforming. In the following, we derive an approximation to the joint density (5) by introducing a perturbation to the matrix dimension. The approximate density function involves only elementary functions that leads to a largest eigenvalue distribution amendable for further analysis.

Consider a perturbation α to ν_2 in (5) and apply the change of variables $\lambda_i = y_i^2/4$, $i = 1, \ldots, r$. The deformed joint density of $\boldsymbol{y} = \{y_1, \ldots, y_r\}$ reads

$$f(\boldsymbol{y}) = \frac{1}{c_\alpha}\Delta\left(\frac{y^2}{4}\right)\left(\prod_{i=1}^{r}\frac{y_i}{2}\right) \times \det\left(G_{0,2}^{2,0}\left(\begin{matrix} - \\ \nu_2 - \alpha,\, \nu_1 + j - 1\end{matrix}\,\middle|\,\frac{y_i^2}{4}\right)\right). \tag{7}$$

The normalizing constant c_α is given by $c_\alpha = \prod_{i=1}^{n}\Gamma(\nu_2 - \alpha + i)\Gamma(\nu_1 + i)\Gamma(i)$, such that $\int_{0\leq y_r\leq\cdots\leq y_1<\infty} f(\boldsymbol{y})\mathrm{d}y_1\cdots\mathrm{d}y_r = 1$, which is a direct application of the Andréief integral in [16] and the determinant identity [17, Eq. (A.18.7)]. Using the definition of Meijer's G-function (6), we have

$$\begin{aligned} &G_{0,2}^{2,0}\left(\begin{matrix} - \\ \nu_2 - \alpha,\, \nu_1 + j - 1\end{matrix}\,\middle|\,\frac{y^2}{4}\right) \\ &= \frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty}\Gamma(\nu_2 - \alpha + z)\Gamma(\nu_1 + j - 1 + z)\left(\frac{y^2}{4}\right)^{-z}\mathrm{d}z. \end{aligned} \tag{8}$$

The integrand in the RHS of (8) has simple poles at $z = -\nu_1 - j + 1 - k$ and $z = -\nu_2 + \alpha - k$, for $k = 0, 1, 2, \ldots$, and the integration path can be selected as a vertical line in the complex plane with $c > \max(-\nu_1 - j + 1, -\nu_2 + \alpha)$. Since the residue of Gamma function equals $\mathrm{Res}_{z=-k}\,\Gamma(z) = (-1)^k/k!$ at $z = -k$, by the residue theorem, (8) becomes

$$\frac{1}{y^{\theta_j}}\left(\frac{y}{2}\right)^{\psi_j}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\alpha - k)_{\theta_j}\Gamma(\alpha - k)\frac{y^{2k-2\alpha+1}}{2^{2k-\theta_j}} + \sum_{k=0}^{\infty}\frac{(-1)^{k+\theta_j}}{k!}\frac{\Gamma(-k-\alpha)}{(\alpha + k + 1)_{\theta_j}}\frac{y^{2k+1+2\theta_j}}{2^{2k+1+\theta_j}}\right), \tag{9}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol, $\theta_j = \nu_1 - \nu_2 + j - 1$, and $\psi_j = \nu_1 + \nu_2 + j - 2$.

By setting the perturbation $\alpha = 1/2$ and using the identity

[2] $(\cdot)^\dagger$ denotes the conjugate transpose operation.

[3] In this paper, the dimensions of matrices in the determinants are $r \times r$, i.e., $i, j = 1, \ldots, r$, unless otherwise stated.

$$G^{m,n}_{p,q}\left(\begin{array}{c} a_1,\ldots,a_n,a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q \end{array}\middle| x\right) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+z)\prod_{j=1}^{n}\Gamma(1-a_j-z)}{\prod_{j=n+1}^{p}\Gamma(a_j+z)\prod_{j=m+1}^{q}\Gamma(1-b_j-z)} x^{-z}\,\mathrm{d}z. \quad (6)$$

$\Gamma(1/2-k) = (-2)^k\sqrt{\pi}/\prod_{i=1}^{k}(2i-1)$, (9) can be written as

$$\frac{\sqrt{\pi}}{y^{\theta_j}}\left(\frac{y}{2}\right)^{\psi_j}\left(\sum_{i\text{ is even}}\frac{y^i}{i!}2^{\theta_j}\left(\frac{1-i}{2}\right)_{\theta_j} - \sum_{i\text{ is odd}}\frac{y^{i+2\theta_j}}{(i+2\theta_j)!}\left(-\frac{1}{2}\right)^{\theta_j}\frac{(i+2\theta_j)!}{i!(1+i/2)_{\theta_j}}\right). \quad (10)$$

According to the definition of Pochhammer symbol, the coefficients for the even power of y in the series (10) equal

$$2^{\theta_j}\left(\frac{1-i}{2}\right)_{\theta_j} = \prod_{k=1}^{\theta_j}(2k-1-i) = A(i). \quad (11)$$

Due to [18, Eq. (4.2.7.14)], $A(i)$ can be expressed in the form of a finite series as

$$A(i) = \sum_{k=0}^{\theta_j}(-1)^k\frac{i!(2\theta_j-k)!2^{k-\theta_j}}{(i-k)!(\theta_j-k)!k!}. \quad (12)$$

Similarly, the coefficients for the odd power of y in (10) can be calculated as

$$\frac{(-1)^{\theta_j}(i+2\theta_j)!}{2^{\theta_j}i!(1+i/2)_{\theta_j}} = \prod_{k=1}^{\theta_j}(1-2k-i) = A(i+2\theta_j), \quad (13)$$

where the second equality is obtained by rearranging the order of product such that $k=\theta_j-l+1$. Moreover, it is clear from (13) that $A(i)=0$ when $i=1,3,\ldots,2\theta_j-1$. Therefore, by substituting (11)–(13) into (10), the Meijer's G-function (8) is simplified to (15) on the top of the next page, where $P_j(y)$ denotes a θ_j-th degree polynomial

$$P_j(y) = \sum_{k=0}^{\theta_j}\frac{(2\theta_j-k)!2^{k-\theta_j}}{(\theta_j-k)!k!}y^k. \quad (16)$$

In (14), the summation over k is truncated at $\min(\theta_j,i)$, since the factorial is infinity at negative integer numbers. We obtain (15) by changing the order of summation with $l=i-k$ in (14). Substituting (15) into (7), the deformed joint density function $f(\boldsymbol{y})$ with $\alpha=1/2$ is given by

$$f(\boldsymbol{y}) = \frac{1}{c'_\alpha}\Delta(y^2)\left(\prod_{i=1}^{r}y_i^{2\nu_2}e^{-y_i}\right)\det\left(P_j(y_i)\right), \quad (17)$$

where the normalization factor c'_α is

$$c'_\alpha = \frac{2^{3/2r^2+(\nu_1+\nu_2-3/2)r}}{\pi^{r/2}}c_\alpha. \quad (18)$$

C. Largest Eigenvalue Distribution

The marginal distribution of $y_1 = \max_{1\le i\le r}(y_i)$ is obtained via the joint density (17), such that $F_{Y_1}(y) = \int_{\mathcal{D}}f(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$, where $\mathcal{D}$ denotes the support $\{0\le y_n\le\ldots\le y_1\le y\}$. Using Andréief integral in [16], we have

$$F_{Y_1}(y) = \frac{1}{c'_\alpha}\det\left(\sum_{k=0}^{\theta_j}\frac{(2\theta_j-k)!2^{k-\theta_j}}{(\theta_j-k)!k!}\gamma(a_{i,k},y)\right), \quad (19)$$

where $a_{i,k} = 2\nu_2+2i+k-1$ and $\gamma(a_{i,k},y)$ denotes the lower incomplete Gamma function. Since $a_{i,k}$ is a positive integer, $\gamma(a_{i,k},y)$ can be explicitly written according to [19, Eq. (8.352.1)] as

$$\gamma(a_{i,k},y) = \Gamma(a_{i,k})\left(1-e^{-y}\sum_{m=0}^{a_{i,k}-1}\frac{y^m}{m!}\right). \quad (20)$$

By careful examination of the entries of determinants in (19), it can be shown that the distribution function $F_{Y_1}(y)$ admit the following form

$$F_{Y_1}(y) = 1+\sum_{p=1}^{r}e^{-p\,y}\sum_{q=0}^{q_m}c_{p,q}\,y^q, \quad (21)$$

where

$$q_m = \left(r+s+t-\frac{1}{2}\right)p-\frac{5}{2}p^2, \quad (22)$$

and $c_{p,q}$ denotes unknown coefficient depending on the matrix dimensions (r,s,t). Note that the expression (21) is similar to the largest eigenvalue distribution of the Wishart matrix derived as [2, Eq. (22)], where the unknown coefficients can be computed numerically according to [3]. Using a similar algorithm, one can tabulate the coefficient $c_{p,q}$ for arbitrary matrix dimensions (r,s,t) and the resulting expression (21) enables further algebraic manipulations.

We hereafter use the notations $F_{Y_1}(y,t)$, $c_{i,j}(t)$, and $q_m(t)$ to specify the dependence of $F_{Y_1}(y)$, $c_{i,j}$, and q_m on the matrix dimension t. With a change of variable $y=2\sqrt{x}$ in (21), we observed in the numerical simulations that $F_{Y_1}(2\sqrt{x},t)$ and $F_{Y_1}(2\sqrt{x},t+1)$ are approximately vertical-shifted versions of the largest eigenvalue distribution $F_{\lambda_1}(x)$. By a simple interpolation between $F_{Y_1}(2\sqrt{x},t)$ and $F_{Y_1}(2\sqrt{x},t+1)$, an approximation to $F_{\lambda_1}(x)$ can be constructed as

$$\begin{aligned} F_{\lambda_1}(x) &\approx \frac{1}{2}\left(F_{Y_1}(2\sqrt{x},t)+F_{Y_1}(2\sqrt{x},t+1)\right) \\ &= 1+\sum_{i=1}^{r}e^{-2i\sqrt{x}}\sum_{j=0}^{q_m(t+1)}\mathcal{C}_{i,j}\,x^{j/2}, \end{aligned} \quad (23)$$

where $\mathcal{C}_{i,j} = 2^{j-1}\left(c_{i,j}(t)+c_{i,j}(t+1)\right)$ and we define $c_{i,j}(t)=0$ when $j>q_m(t)$.

IV. Performance Analysis

A. Ergodic Mutual Information

The ergodic MI of the MIMO beamforming system over Rayleigh product channels is given by

$$C(\gamma) = \int_0^\infty\log\left(1+\gamma\frac{x}{n_1}\right)\mathrm{d}F_{\lambda_1}(x). \quad (24)$$

$$
G_{0,2}^{2,0}\left(\begin{matrix} - \\ \nu_2 - \alpha,\, \nu_1 + j - 1 \end{matrix} \,\middle|\, \frac{y^2}{4} \right) = \frac{\sqrt{\pi}}{y^{\theta_j}} \left(\frac{y}{2}\right)^{\psi_j} \sum_{i=0}^{\infty} \frac{(-y)^i}{i!} A(i) = \frac{\sqrt{\pi}}{y^{\theta_j}} \left(\frac{y}{2}\right)^{\psi_j} \sum_{i=0}^{\infty} \sum_{k=0}^{\min(\theta_j, i)} \frac{(-1)^{i+k} y^i (2\theta_j - k)! 2^{k-\theta_j}}{(i-k)!(\theta_j - k)! k!} \tag{14}
$$

$$
= \frac{\sqrt{\pi}}{y^{\theta_j}} \left(\frac{y}{2}\right)^{\psi_j} \left(\sum_{l=0}^{\infty} \frac{(-y)^l}{l!} \right) \sum_{k=0}^{\theta_j} \frac{(2\theta_j - k)! 2^{k-\theta_j}}{(\theta_j - k)! k!} y^k = \frac{\sqrt{\pi}}{y^{\theta_j}} \left(\frac{y}{2}\right)^{\psi_j} e^{-y} P_j(y) \tag{15}
$$

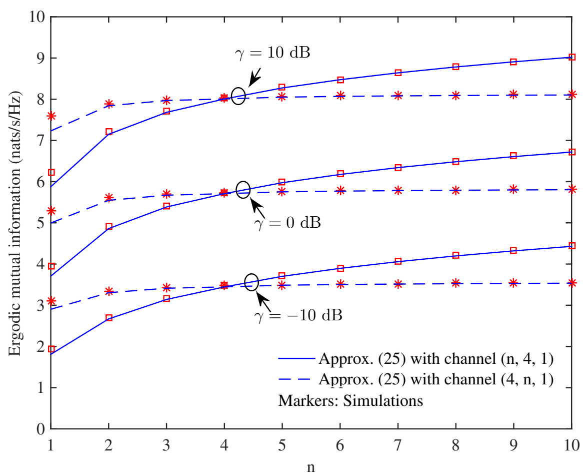

Fig. 1. Ergodic mutual information of MIMO beamforming over Rayleigh product channels. Solid line with '□': increase transmit antennas while keeping $n_1 = 4$ and $n_2 = 1$; dashed line with '$*$': increase scattering objects while keeping $n_0 = 4$ and $n_2 = 1$; markers: simulations.

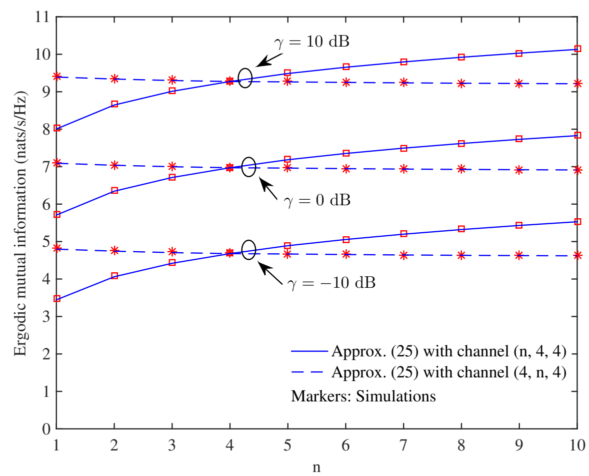

Fig. 2. Ergodic mutual information of MIMO beamforming over Rayleigh product channels. Solid line with '□': increase transmit antennas while keeping $n_1 = n_2 = 4$; dashed line with '$*$': increase scattering objects while keeping $n_0 = n_2 = 4$; markers: simulations.

Using integration by parts and inserting the approximation (23) into (24), we obtain

$$
\begin{aligned}
C(\gamma) &= \int_0^{\infty} \left(1 - F_{\lambda_1}(x)\right) \frac{\rho}{1+\rho x} \mathrm{d}x \\
&\approx -\frac{1}{\sqrt{\pi}} \sum_{i=1}^{r} \sum_{j=0}^{q_m(t+1)} \frac{\mathcal{C}_{i,j}}{\rho^{j/2}} G_{1,3}^{3,1}\left(\begin{matrix} -j/2 \\ -j/2,\, 0,\, 1/2 \end{matrix} \,\middle|\, \frac{i^2}{\rho} \right),
\end{aligned} \tag{25}
$$

where $\rho = \gamma/n_1$ and (25) is obtained by using the identities [19, Eqs. (3.389.6) and (9.34.6)].

Figs. 1 and 2 show the ergodic MI of MIMO beamforming over various Rayleigh product channels assuming the SNR $\gamma = -10, 0, \text{and}\, 10$ dB, respectively. In Fig. 1, we plot the proposed approximation (25) as a function of the number of transmit antennas n_0, while fixing the scattering objects $n_1 = 4$ and the receive antennas $n_2 = 1$, i.e., $(n_0, 4, 1)$. As a comparison, the ergodic MI of the channels $(4, n_1, 1)$ is plotted while increasing the number of scatterings n_1. At the considered SNRs, it is clear that the performance improvement is more significant by using more transmit antennas. When the receiver is equipped with multiple antennas, Fig. 2 shows that the performance of MIMO beamforming degrades as the transceivers see more scatterings. This is due to the fact that power of the MIMO channel disperses over multiple eigenchannels. At a fixed array gain $n_0 n_2$, the power of the dominant eigenchannel, corresponding to λ_1, decreases as the rank of matrix increases.

B. Average Symbol Error Rate

For a wide variety of modulation schemes, the average SER of MIMO beamforming systems can be obtained by integrating over the instantaneous SNR (4) as

$$
\mathrm{SER}(\gamma) = a\, \mathbb{E}\left[Q\left(\sqrt{2b\rho\, \lambda_1} \right) \right], \tag{26}
$$

where the expectation is over λ_1 and $Q(x) = 1/\sqrt{2\pi} \int_x^{\infty} e^{-u^2/2} \mathrm{d}u$ is the Gaussian Q-function. The parameters a and b are modulation specific constants ([20]). For instance, $a = 2(M-1)/M$ and $b = 3/(M^2-1)$ for the M-ary Pulse Amplitude Modulation (PAM), $a = 2$ and $b = \sin^2(\pi/M)$ for the M-ary Phase-Shift Keying (PSK), and $a = 4$ and $b = 3/(2M-2)$ for the M-ary Quadrature Amplitude Modulation (QAM). Note that the expression (26) is an approximation in cases of PSK and QAM modulations. By substituting (23) into (26) and applying integration by parts, we have

$$
\begin{aligned}
\mathrm{SER}(\gamma) &= \frac{a}{2} \sqrt{\frac{b\rho}{\pi}} \int_0^{\infty} F_{\lambda_1}(x) \frac{e^{-b\rho x}}{\sqrt{x}} \mathrm{d}x \\
&\approx \frac{a}{2} + \frac{a}{2} \sqrt{\frac{b\rho}{\pi}} \sum_{i=1}^{r} \sum_{j=0}^{q_m(t+1)} \mathcal{C}_{i,j} L_{i,j}(b\rho),
\end{aligned} \tag{27}
$$

where

$$
L_{i,j}(b\rho) = \int_0^{\infty} e^{-b\rho x - 2i\sqrt{x}} x^{(j-1)/2} \mathrm{d}x. \tag{28}
$$

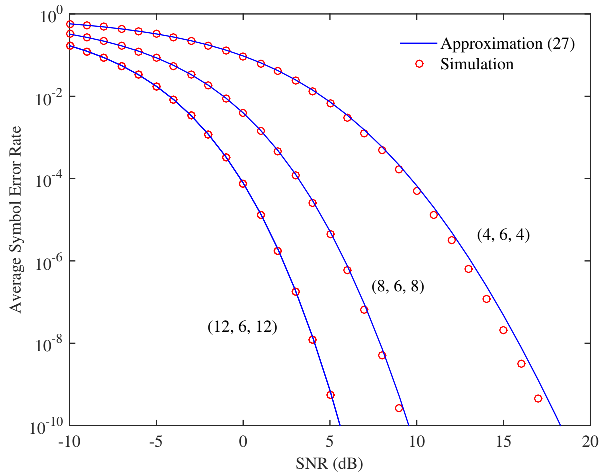

Fig. 3. Average Symbol Error Rate of MIMO beamforming over Rayleigh product channels. Solid line: approximation (27); markers: simulations.

Applying [19, Eq. (3.462.1)], we obtain

$$L_{i,j}(b\rho) = 2(2b\rho)^{-\frac{j+1}{2}}\Gamma(j+1)e^{\frac{i^2}{2b\rho}}D_{-j-1}\left(\sqrt{\frac{2}{b\rho}}i\right), \quad (29)$$

where $D_\nu(\cdot)$ denotes the parabolic cylinder function defined as [19, Eq. (9.240)].

In Fig. 3, the average SER of MIMO beamforming is plotted as a function of the received SNR γ. We compare the approximation (27) with simulated curves, where each curve is obtained by averaging over 10^6 channel realizations. The number of antennas at both transmitter and receiver is set to be $4,\ 8,\ \text{and}\ 12$ while fixing the number of scatterings to be 6. Fig. 3 shows that the proposed approximation achieves a good agreement with the simulation, especially when the number of antennas is large.

V. Conclusions

We studied the performance of MIMO beamforming over the Rayleigh product MIMO channels. By introducing a perturbation to the joint eigenvalue density of the product Gaussian matrices, we obtained a simple yet accurate approximation to the largest eigenvalue distribution. Based on this result, the ergodic mutual information and average SER are calculated in closed-form for an arbitrary SNR and any channel configuration. Numerical results show that it is more effective to improve the performance of MIMO beamforming by using more transmit antennas than by enriching the scattering environment. When multiple antennas are equipped at the receiver, the performance of MIMO beamforming degrades as the transceivers see more scatterings.

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