Homework # 1: Probability Refresher

Rules:
1. Due 11:30am, Tuesday, February 14, 2012, in boxes next to room 219 Phillips Hall
2. Solve all the 10 problems below for 100 points - **independent work only.**
3. Show full solution to the problems - do not skip steps.

Problem #1:

In a country with $n+1$ citizens, a citizen tells a rumor to another citizen, who, in turn, tells the rumor to a third citizen, etc. At each "step" the recipient of the rumor is chosen at random from the population of the remaining citizens. Calculate the probability that a rumor will be told r times without:

(a) returning to the originator
(b) being repeated by any citizen

Repeat the problem when each time a rumor is told by one person to a gathering of N randomly chosen citizens.

Problem #2:

Bob and Alice start a phone conversation each, at the same time. The length of Bob phone calls is known to be exponentially distributed with average of 3 [min] and Alice phone calls are know to be exponentially distributed with average of 5 [min]. Calculate the probability that:

(a) Alice finishes her phone call before Bob
(b) Bob finishes his phone call before Alice, after the phone calls were already 2 [min] in progress
(c) Bob's phone call will last at least 2 [min] longer than Alice's
(d) The sum of the two calls will be less than 10 [min].

Assume that the lengths of two calls are independent.

Problem #3:

A *repetition code*, which is used to increase the probability of a correct reception of a message over a noisy channel, retransmits the message n times, and uses a *majority voter* for decoding. (Majority voter operates on the bit-by-bit basis; i.e., for every bit, the scheme chooses the value of the bit as the one that appears the most times in the received replicas of the message.)

Assume first that the message contains one bit and that the probability of a correct reception of the bit in a single transmission over the channel is p. Assume that n is odd and that $n=2m-1$, where $m=1,2,3, \ldots$ Calculate the probability of the bit-long message being received in error, as a function of m. Repeat for an arbitrary message length of b [bits].

Problem #4:

In an urn, there are 80 objects of two kinds: *cubes (C)* and *balls (B)*. An object can be either *red (R)* or *green (G).* Note that all the four combinations are possible and that the number of cubes is not necessarily equal to the number of balls. Similarly, the number of red objects is not necessarily equal to the number of green objects. Someone tells us that in the urn there are 20 red cubes, 50 balls, and 30 red objects. An object is randomly selected from the urn.

(a). What is the probability that a green ball is selected?
(b). If we know that a cube has been selected, what is the probability that it's red?
(c). If we know that a red object has been selected, what is the probability that it's a cube?

Problem #5:

Two trains are scheduled to arrive independently at a train station at 12:00noon. The trains can be delayed, but never arrive early. After arriving, a train waits for 15 minutes before leaving. The delay of train *X* is uniformly distributed between 0 and 3 hours. The train *Y* arrives with no delay with probability 0.5 or its delay is uniformly distributed between 0 and 2 hours.

Calculate the probability that:

(a) the train *X* arrives before the train *Y* at the station
(b) the trains meet at the train station
(c) the trains meet at the train station, given that the train *Y* arrives before the train *X*

Problem #6:

A reliable multiprocessor computing system is to be designed, where it is required that at least two processors are available with probability of at least 95%. A processor with average reliability of 60% costs $10K. More reliable processors, with improvement in steps of 10% in reliability, are offered at the cost of $8K per 10% reliability improvement step. Note: all the processors in the system have to be of the same reliability.

Calculate the number of processors and the reliability of each processor that minimizes the total cost of the system. Specify this minimum cost.

Problem #7:

The height of people is modeled as a *normal random variable*. Assume that the height in inches of some population is modeled as a *normal random variable* with average of 68 inches and standard deviation of 5 inches.

Find:

(1) the probability that a (randomly selected) person is taller than 6 feet
(2) the probability that the height of a (randomly selected) person is between 64 and 72 inches

Problem #8:

X and ***Y*** are two *i.i.d* *normal* random variables with zero mean and variance of σ^2.
Define the following transformation:

$$\boldsymbol{R} = \sqrt{\boldsymbol{X}^2 + \boldsymbol{Y}^2} \quad \text{and} \quad \boldsymbol{\Theta} = \tan^{-1}(\boldsymbol{X}/\boldsymbol{Y}), \qquad \text{where } -\pi < \theta < \pi .$$

(a) Find the joint pdf of ***R*** and ***Θ***, $f_{\boldsymbol{R\Theta}}(r,\theta)$.

(b) Find the pdf of ***R***, $f_{\boldsymbol{R}}(r)$, and the pdf of ***Θ***, $f_{\boldsymbol{\Theta}}(\theta)$.

Find the conditional pdf of ***R*** given ***Θ***, $f_{\boldsymbol{R|\Theta}}(r\,|\,\theta)$.

Problem #9:

Prove that the sum of *K* independent Poisson processes with rates $\lambda_i, i = 1\ldots K$ is also Poisson with rate of $\sum_{i=1}^{K} \lambda_i$.

Problem #10:

In the town of Math, the weather is either "rainy" or "sunny." The weather on any day depends only on the weather of the previous day: if it's sunny on one day, there're 75% chances that it will be sunny the following day and 25% chances that it will rain; if it's rainy on one day, there're 50% chances that it will rain next day and 50% chances that it will be sunny. What's the % of sunny days in the town of Math?