ENERGY BASED SYNTHESIS OF TENSION MODULATION IN MEMBRANES

Riccardo Marogna,
Dep. of Information Engineering,
University of Padova
Padova, Italy
marognar@dei.unipd.it

Federico Avanzini,
Dep. of Information Engineering,
University of Padova
Padova, Italy
avanzini@dei.unipd.it

Balázs Bank,
Dep. of Measurement and Information Systems,
Budapest University of Technology and Economics
Budapest, Hungary
bank@mit.bme.hu

ABSTRACT

Above a certain amplitude, membrane vibration becomes nonlinear due to the variation of surface tension. This leads to audible pitch glides, which greatly contribute to the characteristic timbre of tom-tom drums of the classical drum set and many other percussion instruments. Therefore, there is a strong motivation to take the tension modulation effect into account in drum synthesis. Some models do already exist that model this phenomenon, however, their computational complexity is significantly higher compared to linear membrane models. This paper applies an efficient methodology previously developed for the string to model the quasistatic part (short-time average) of the surface tension. The efficient modeling is based on the linear relationship between the quasistatic tension and membrane energy, since the energy can be computed at a relatively low computational cost. When this energy-based tension modulation is added to linear membrane models, the perceptually most relevant pitch glides are accurately synthesized, while the increase in computational complexity is negligible.

1. INTRODUCTION

While the most relevant features of membrane oscillation can be conveniently described by the 2-D linear wave equation, some important secondary effects can only be accounted for by introducing nonlinear terms in the equation. Specifically, above a certain amplitude of vibration, the assumption of constant membrane area does not hold, and the tension varies in dependence of the instantaneous displacement. Because this nonlinearity comes from the geometry of the problem (the elasticity of the membrane material is assumed to be linear), it is called "geometric nonlinearity".

For strings, the effects of the geometric nonlinearity can be classified into different regimes, depending on string parameters and on the excitation force [1]. One important special case is tension modulation, where the tension varies in time but it is spatially uniform along the string. The most important perceptual effect of tension modulation is the pitch glide, meaning that the pitch of the string decreases as the sound decays. Relevant cases where this effect can be audible include electric and steel-stringed acoustic guitars, and various ethnic instruments.

The effects of the geometric nonlinearity for the case of the membrane are somewhat less studied, and there appears to be no proposed classification of the oscillation into different regimes. However, it is known that the pitch glides coming from tension modulation have an even more significant perceptual effect in membranes than in strings. As an example, tom-toms in a drum set exhibit a characteristic glide at medium-high dynamic ranges, but many other percussion instruments produce this effect. Therefore, there is a strong motivation to simulate tension modulation in physics-based membrane synthesis.

Membrane models proposed in previous works are mainly based on 2-D or 3-D digital waveguide meshes (DWM [2]), which can provide accurate simulation of wave propagation, depending on the mesh topology [3], and with additional processing to compensate for dispersion [4]. Finite-difference schemes have also been used [5] (see [6] for an analysis of various schemes). Models based on modal synthesis [7] also exist. Cook [8] proposed a series of "physically-informed" approaches to the modeling of percussion sounds, including modal synthesis. Rabenstein and coworkers have applied the functional transformation method to the simulation of rectangular and circular linear membranes [9].

Models for tension modulation in strings have been discussed for waveguide [10] and modal [11, 12] approaches. An extension to the 2-D case of rectangular membranes has been proposed in [13] in the context of modal synthesis. More recently, a tension-modulated circular membrane model was proposed in [14, 15], based on the theory of vibrations of elastic plates [16]. A finite-difference tension-modulated membrane model has been proposed in [17].

Even for the simplest models of tension modulation in strings and membranes, the computational complexity is significantly higher compared to efficient linear models. As an example, the model proposed in [14] requires the nonlinear tension component to be computed at sample rate from a weighted sum of the squared modal displacements (in the order of several hundreds, or thousands). In [18], a new tension modulation methodology was proposed for the case of the string. The method amounts to approximating the quasistatic part (i.e., the short-time average) of the string tension from the energy of the string. It has been shown in [18] that the method leads to significant computational savings, while it is still able to simulate the perceptually most relevant effect of tension modulation.

The purpose of this paper is to demonstrate that the approach presented in [18] can be extended to the case of the circular membrane. To this end, the same line of reasoning developed for the string is applied here to the membrane. In particular, it is shown that the quasistatic nonlinear tension can be estimated from the total energy of the membrane.

The remainder of the paper is organized as follows. Section 2 summarizes the tension-modulation model presented in [14] and introduces the concept of quasistatic tension in the circular membrane. Section 3 defines kinetic and potential energies of a circular membrane, and relates the total membrane energy to the tension. Section 4 demonstrates how the proposed approach can be exploited to obtain efficient simulations of tension-modulated circular membranes, in the framework of modal synthesis. Finally, Sec. 5 concludes the paper and gives future research directions.

2. TENSION MODULATION IN CIRCULAR MEMBRANES

2.1. General formulation

The model described here is derived in the assumption of homogeneous and isotropic membrane material, and of uniform clamping (see [14, 15] for more details). The membrane vertical displacement $z(r,\varphi,t)$, driven by a force density $f^{(\text{ext})}(r,\varphi,t)$, is governed by the following equation:

$$D\nabla^4 z + \sigma\frac{\partial^2 z}{\partial t^2} - [T_0 + T_{NL}(z)]\nabla^2 z + + d_1\frac{\partial z}{\partial t} + d_3\frac{\partial \nabla^2 z}{\partial t} = f^{(\text{ext})}, \quad (1)$$

Table 1: *Physical and geometrical membrane parameters.*

Symbol	Unit	Meaning
σ	Kg/m^2	Surface density
T_0	N/m	Surface tension
d_1	Kg/sm^2	Freq. independent dissipation coefficient
d_3	Kg/sm	Freq. dependent dissipation coefficient
E	N/m^2	Young modulus
υ	—	Poisson ratio
R	m	Radius
h	m	Thickness

where we have omitted spatial and temporal dependencies for simplicity. The units and meanings of all physical parameters in Eq. (1) are listed in Table 1. The coefficient $D = Eh^3/12(1-\upsilon^2)$ is the (small) bending stiffness of the membrane. The function $T_{NL}(z)$ can be interpreted as the surface tension generated in dependence of the displacement z, in addition to the tension at rest T_0. Ideal boundary conditions for a uniformly-clamped membrane are given by zero deflection and skewness at the boundary.

A model for the term $T_{NL}(z)$ is derived from the theory of elastic plates. Specifically, it is based on the so-called Berger approximation of the von Karman equations for thin plates subjected to lateral and in-plane forces [16]:

$$T_{NL}(z) = \frac{Eh}{2\pi R^2(1-\upsilon^2)} \cdot \int_0^R\int_0^{2\pi}\left[\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \varphi}\right)^2\right] r\,d\varphi\,dr. \quad (2)$$

The double integral in Eq. (2) computes the difference between the membrane area $A(z)$ corresponding to the displacement z, and the area at rest $A_0 = \pi R^2$. Accordingly, the nonlinear function $T_{NL}(z)$ can be interpreted as a spatially uniform tension modulation term, which depends only on the total area $A(z)$, in analogy with the Kirchhoff-Carrier equation for tension-modulated strings [19].

2.2. Modal formulation

The general solution z can be expressed in terms of its normal modes $\bar{z}(t)K(r,\varphi)$ in which temporal and spatial dependencies are decoupled. With the boundary conditions considered here, the equation is known [20] to have a numerable set of modes with spatial eigenfunctions:

$$K_{n,m}(r,\varphi) = \cos[n(\varphi-\varphi_0)]J_n\left(\mu_{n,m}\frac{r}{R}\right), \quad (3)$$

where $n \geq 0$, $m \geq 1$, and $\mu_{n,m}$ is the m-th zero of the n-th order Bessel function of the first kind, J_n.

The PDE in (1) can then be turned into a set of ordinary differential equations that describe the dynamics of the normal modes. More precisely, the mode (n, m), characterized by the modal amplitude $\bar{z}_{n,m}$, is a forced second-order oscillator whose parameters are determined by those of the original PDE. The displacement z and the normal modes are related through the following equations:

$$\bar{z}_{n,m}(t) = \int_r \int_\varphi z(r, \varphi, t) K_{n,m}(r, \varphi)\; r dr d\varphi. \tag{4a}$$

$$z(r, \varphi, t) = \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} \frac{\bar{z}_{n,m}(t) K_{n,m}(r, \varphi)}{\| K_{n,m}(r, \varphi) \|_2^2}, \tag{4b}$$

where Eq. (4a) expresses the displacement as a weighted sum of its modal amplitudes. It was shown in [14] that $\bar{T}_{NL}$ can also be written in terms of the modal amplitudes $\bar{z}$:

$$\bar{T}_{NL}(\bar{z}) = \frac{Eh}{2\pi R^4(1-\upsilon^2)} \cdot \sum_{n,m} \frac{\mu_{n,m}^2 \bar{z}_{n,m}^2}{\| K_{n,m} \|_2^2}. \tag{5}$$

By means of this nonlinear equation the model (1) can be integrated into a modal synthesis engine.

2.3. Quasistatic tension modulation

In this section we show that the nonlinear tension term can be split into a "quasistatic tension" component and a second component containing "double-frequency terms". The derivation resembles closely that for the string [18].

After the excitation, the membrane modes decay exponentially, thus the instantaneous amplitudes $\bar{z}_{n,m}(t)$ become exponentially decaying sinusoidal functions:

$$\bar{z}_{n,m}(t) = A_{n,m} \sin(\omega_{n,m} t + \phi_{n,m}) e^{-t/\tau_{n,m}}, \tag{6}$$

where amplitudes $A_{n,m}$ and phases $\phi_{n,m}$ depend on the initial conditions, while resonance frequencies $\omega_{n,m}$ and decay times $\tau_{n,m}$ are determined as follows, from [14]:

$$\omega_{n,m}^2 = \left(\frac{\mu_{n,m}}{R}\right)^2 \left[\frac{D}{\sigma}\left(\frac{\mu_{n,m}}{R}\right)^2 + c^2\right], \tag{7a}$$

$$\tau_{n,m}^{-1} = \frac{1}{2\sigma}\left[d_1 + d_3\left(\frac{\mu_{n,m}}{R}\right)^2\right], \tag{7b}$$

and $c = \sqrt{T_0/\sigma}$ is the wave velocity in the membrane.

Substituting this expression into the nonlinear tension term (5) yields

$$T_{NL}(t) = \frac{Eh}{4\pi R^4(1-\upsilon^2)} \sum_{n,m} \frac{\mu_{n,m}^2 A_{n,m}^2}{\| K_{n,m} \|_2^2} \cdot \left[1 - \cos(2\omega_{n,m} t + 2\phi_{n,m})\right] e^{-2t/\tau_{n,m}}. \tag{8}$$

The first time-dependent part of Eq. (8) is a quasistatic increase of tension:

$$T_{qs}(t) = \frac{Eh}{4\pi R^4(1-\upsilon^2)} \sum_{n,m} \frac{\mu_{n,m}^2 A_{n,m}^2}{\| K_{n,m} \|_2^2} \cdot e^{-2t/\tau_{n,m}}, \tag{9}$$

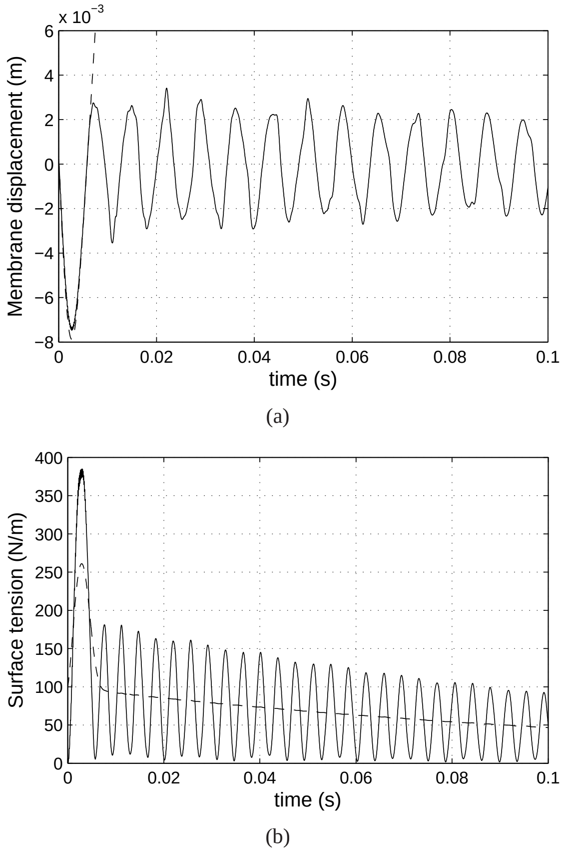

Figure 1: *Simulated membrane excited at the center with impact velocity of* 5 *m/s; (a) membrane displacement (solid line) and hammer displacement (dashed line) at the impact position; (b) nonlinear tension* T_{NL} *(solid line) versus quasistatic component* T_{qs} *(dashed line).*

which decays slowly. This leads to a proportional shift (continuous decrease) in the modal frequencies, resulting in a pitch glide which is the most relevant perceptual effect of tension modulation.

As an example, the vibration of a simulated circular membrane is displayed in Fig. 1. The example is computed using modal synthesis, and the membrane is excited by a nonlinear impact force model (see [14] and Sec. 4 below). In this example the membrane has been struck near the center (at a point $r_h = 0.1R$) with a moderate impact velocity (5 m/s), and the tension at rest is 800 N/m.

Being proportional to the total membrane area, the nonlinear tension $T_{NL}(t)$ (the solid line in Fig. 1(b)) oscillates around the quasistatic tension T_{qs}. Here the quasistatic tension (dashed line in Fig. 1(b)) is estimated by applying a lowpass filter to the tension T_{NL}. The slow initial rise of the quasistatic tension T_{qs} is the time-domain side effect of lowpass filtering.

3. RELATION TO ENERGY

Since the most prominent effect of tension modulation in the membrane is the pitch glide due to the quasistatic tension variation T_{qs}, it is reasonable to concentrate on the modeling of the quasistatic part. In this section, we derive a simple relationship between T_{qs} and the membrane energy. As will be shown later, the energy of the membrane can be estimated at lower computational complexity, thus allowing to compute the quasistatic tension with less operations compared to earlier models.

3.1. Membrane kinetic and potential energies

The kinetic energy E_k and the potential energy E_p of the membrane can be derived by applying the same line of reasoning used in [21, Ch.3-5] for the string and the rectangular membrane.

The kinetic energy associated with an infinitesimal element $dA = rdrd\varphi$ of the membrane area is $dE_k(r,\varphi,t) = \frac{1}{2}\sigma dA\dot{z}^2(r,\varphi,t)$. Therefore, the total kinetic energy is

$$E_k(t) = \frac{1}{2}\sigma \int_r \int_\varphi \left[\frac{\partial z(r,\varphi,t)}{\partial t}\right]^2 rdrd\varphi. \qquad (10)$$

The potential energy E_p can be derived as follows. Suppose that the membrane is displaced from equilibrium to its final displacement z through the intermediate displacements kz (with $k \in [0,1]$). The potential energy equals the work done along this path. Throughout the path, the forces acting on a membrane area element $dA = rdrd\varphi$ are

$$\begin{aligned} F_r(r,\varphi,t,k) &= \frac{T_0}{r}\frac{\partial}{\partial r}\left[r\frac{\partial kz(r,\varphi,t)}{\partial r}\right] rdrd\varphi, \\ F_\varphi(r,\varphi,t,k) &= \frac{T_0}{r^2}\frac{\partial^2 kz(r,\varphi,t)}{\partial\varphi^2} rdrd\varphi, \end{aligned} \qquad (11)$$

along the radial and angular direction, respectively. Therefore, the net force acting on the area element is

$$F(r,\varphi,t,k) = T_0\nabla^2 kz(r,\varphi,t)rdrd\varphi. \qquad (12)$$

The associated potential energy dE_p equals (with the opposite sign) the work done by this force along the path from equilibrium to y. For each increase dk the corresponding change in displacement is zdk, therefore:

$$\begin{aligned} dE_p(r,\varphi,t) &= -\int_0^1 \left[T_0\nabla^2 kz(r,\varphi,t)rdrd\varphi\right] z(r,\varphi,t)dk \\ &= -\frac{T_0}{2}\nabla^2 z(r,\varphi,t)z(r,\varphi,t)rdrd\varphi. \end{aligned} \qquad (13)$$

Therefore, the total potential energy is

$$E_p(t) = -\frac{T_0}{2}\int_r\int_\varphi \nabla^2 z(r,\varphi,t)z(r,\varphi,t)rdrd\varphi. \qquad (14)$$

We have implicitly assumed $E_p = 0$ at equilibrium. Note also that Eq. (14) represents in fact the potential energy in the linear regime, as the forces F_r and F_φ acting on the area element dA have been estimated by only considering T_0 and discarding T_{NL}. Finally, Eq. (14) does not consider bending forces, which are assumed to be negligible (i.e. D is assumed to be small).

The sum of the kinetic and potential energy $E = E_k + E_p$ is the total energy of the membrane, and the membrane at rest leads to $E = 0$.

3.2. Quasistatic tension and total energy

We now show that the quasistatic tension T_{qs} is a scaled version of the total membrane energy E, through Eq. (20).

In order to prove this result, first the energy has to be written in terms of the normal modes. Regarding E_k, substitution of Eq. (4b) into (10) yields

$$\begin{aligned} E_k(t) &= \frac{\sigma}{2}\sum_{n,m}\frac{\dot{z}^2_{n,m}(t)}{\parallel K_{n,m}\parallel_2^4}\int_r\int_\varphi K^2_{n,m}(r,\varphi)rdrd\varphi = \\ &= \frac{\sigma}{2}\sum_{n,m}\frac{\dot{z}^2_{n,m}(t)}{\parallel K_{n,m}\parallel_2^2}. \end{aligned} \qquad (15)$$

Moreover, substitution of Eq. (6) into (15) yields[1]

$$E_k(t) = \frac{\sigma}{4}\sum_{n,m}\frac{A^2_{n,m}\omega^2_{n,m}}{\parallel K_{n,m}\parallel_2^2}[1+\cos(2\omega_{n,m}t+2\phi_{n,m})]e^{-2t/\tau_{n,m}}. \qquad (16)$$

The potential energy E_p can be written in terms of the normal modes by substituting Eq. (4b) into (14). Recalling the the modal shapes $K_{n,m}$ are eigenfunctions of the Laplacian operator with associated eigenvalues $-(\mu_{nm}/R)^2$, and that they are orthogonal, one finds

$$\begin{aligned} E_p(t) &= -\frac{T_0}{2}\int_r\int_\varphi\left[\sum_{n,m} -\frac{\mu^2_{n,m}}{R^2}\frac{z^2_{n,m}(t)}{\parallel K_{n,m}\parallel_2^4}K^2_{n,m}(r,\varphi)\right]rdrd\varphi \\ &= \frac{T_0}{2R^2}\sum_{n,m}\frac{\mu^2_{n,m}}{\parallel K_{n,m}\parallel_2^2}z^2_{n,m}(t). \end{aligned} \qquad (17)$$

Moreover, substitution of Eq. (6) into (17) yields

$$E_p(t) = \frac{T_0}{4R^2}\sum_{n,m}\frac{\mu^2_{n,m}A^2_{n,m}}{\parallel K_{n,m}\parallel_2^2}[1-\cos(2\omega_{n,m}t+2\phi_{n,m})]e^{-2t/\tau_{n,m}}. \qquad (18)$$

Finally, recalling the identity $\omega^2_{n,m} = \frac{\mu^2_{n,m}T_0}{R^2\sigma}$ (from Eq. (7a) in the hypothesis of negligible bending stiffness, $D \sim 0$),

[1] Here the derivatives $\dot{z}^2_{n,m}(t)$ are approximated in the assumption that the decay times $\tau_{n,m}$ are long compared to the oscillation periods $2\pi/\omega_{n,m}$. This assumption is usually taken when deriving the kinetic energy of a second-order damped oscillator [21, Ch.2].

the total energy can be written as

$$E(t) = E_k(t) + E_p(t) = \frac{T_0}{2R^2} \sum_{n,m} \frac{\mu_{n,m}^2 A_{n,m}^2}{\| K_{n,m} \|_2^2} e^{-2t/\tau_{n,m}}. \tag{19}$$

Comparison of this equation with Eq. (9) proves that

$$T_{qs}(t) = \frac{Eh}{2\pi R^2 (1-\upsilon^2) T_0} E(t), \tag{20}$$

which is the fundamental outcome of this section.

4. SIMULATIONS

In this section the equations presented above are validated through simulations of a discrete-time realization of the nonlinear membrane model, and an efficient implementation is presented. The model utilizes a modal synthesis approach in which the membrane is modeled by a bank of second-order resonators, each representing the displacement of one mode of oscillation. Tension modulation is estimated from modal displacements according to Eq. (5). The membrane is hit by a "hammer" (a drumstick or a mallet) which is modeled as lumped mass m_h. The impact force F_h between the hammer and the membrane is, from [22]:

$$F_h(\zeta(t), \dot{\zeta}(t)) = \begin{cases} k\zeta(t)^\alpha + \lambda \zeta(t)^\alpha \dot{\zeta}(t), & \zeta > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where $\zeta(t) = z(r_h, \varphi_h, t) - z_h(t)$ represents the hammer compression, i.e. the difference between the hammer displacement z_h and the displacement of the membrane z at the point of contact (r_h, φ_h). Similar nonlinear nearly elastic models have been used to describe the interaction of a hammer with piano strings [23], as well as that of a drumstick or mallet and a membrane [24].

4.1. Computation of quasistatic tension from energy

The energy of the same membrane as of Fig. 1 is displayed in Fig. 2(a). The kinetic and potential energies oscillate in antiphase, and as a result the total energy is a slowly decaying signal. Note that the total energy computed from the simulation is not exactly a monotonically decreasing signal, and instead exhibits small oscillations. This is to be attributed to the fact that the potential energy E_p in Eq. (14) has been derived in the linear regime, discarding the effects of tension modulation.

Figure 2(b) shows a comparison of quasistatic tension computed from membrane energy according to Eq. (20), and quasistatic tension computed by lowpass filtering T_{NL}. The energy-based quasistatic tension (solid line) shows the correct behavior (apart from the presence of small ripples already discussed above).

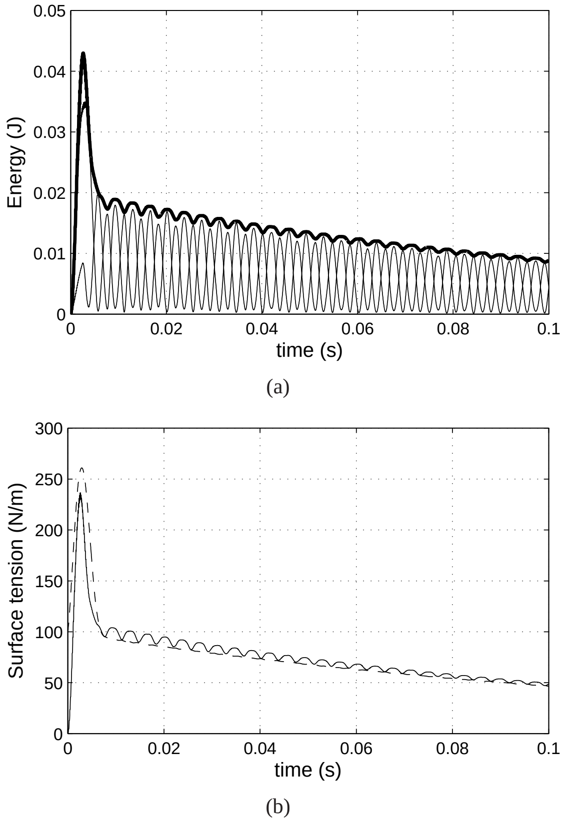

Figure 2: *Simulated struck membrane; (a) total (solid thick line), kinetic, and potential (solid thin lines) energies; (b) quasistatic tension T_{qs} computed from membrane energy (solid line) and estimated from T_{NL} (dashed line).*

These results show that the quasistatic tension can be accurately computed from the membrane energy. In the model discussed in [14, 15], tension computation is a separate model block which acts as an input to the linear filters representing membrane modes. This block can thus be substituted by an energy computation block and a simple scaling.

4.2. Efficient modal synthesis

Substituting the tension modulation block with a scaled energy computation block, as suggested above, will not yet lead to computational savings because computing the membrane energy from the modal displacements (from Eqs. (10) and (14)) takes a similar number of operations as computing the tension. However, the computational complexity of the energy computation can be decreased significantly, as shown in this section.

As can be seen in Fig. 2, the energy and the quasistatic tension are slowly changing signals, in contrast to the total tension variation (see Fig. 1(b), solid line). Therefore, the membrane energy can be computed at a lower rate, and the continuous energy curve can be obtained by linearly interpolating between the computed points. As a result, the average load of energy computation becomes negligible compared to the sample-rate computation of the modal displacements.

Figure 3 shows three spectrograms obtained by hitting the membrane model at a point $r_h = 0.5R$ with impact velocity 20 m/s, which is close to the highest dynamic levels in drum playing [25]. In Fig. 3(a) the complete tension-modulation model T_{NL} of Eq. (5) has been used; the spectrogram in Fig. 3(b) instead has been obtained by approximating T_{NL} with T_{qs}, estimated from the membrane energy according to Eq. (20); finally Fig. 3(c) shows the results obtained when the computation of T_{qs} is downsampled by a factor 32 (i.e. T_{qs} is computed every $32/44.1 \sim 0.73$ ms).

Although some differences can be noticed between the spectrograms, the overall behavior is similar. The same remarks applies to the corresponding sounds: they are perceptually similar, although some differences are heard. The sounds are available on a dedicated webpage[2] for the interested reader.

5. CONCLUSION AND FUTURE WORK

This paper has presented an efficient methodology for the modeling of tension modulation effects in circular membranes, based on the linear relationship between the energy of the membrane and the quasistatic part (short-time average) of the tension variation. In summary, the computationally heavy tension computation block in earlier membrane models is substituted with a more efficient energy computation block and a simple scaling.

As a result, the simplified model is able to synthesize the frequency glides occurring in tension modulated membranes, with little additional computational complexity compared to a linear model, in contrast to earlier tension modulated membrane models.

We expect the approach proposed in this paper to be generalizable to other simple geometries (e.g. rectangular membranes). Moreover, it should be possible to further simplify the computations by applying the “energy storage model” [18] originally developed for the string. Finally, future research includes thorough comparison between the proposed simplified model and earlier models, in terms of sound quality and computational complexity.

6. ACKNOWLEDGMENTS

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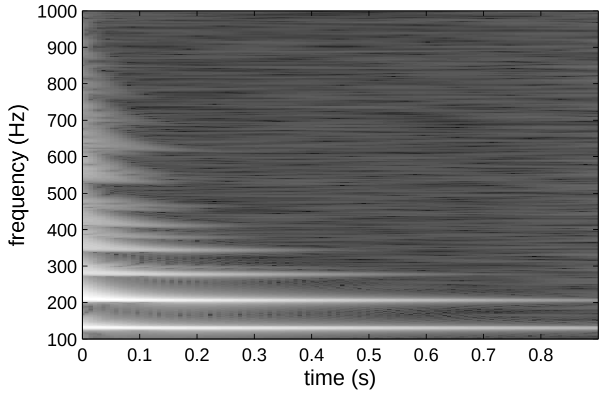
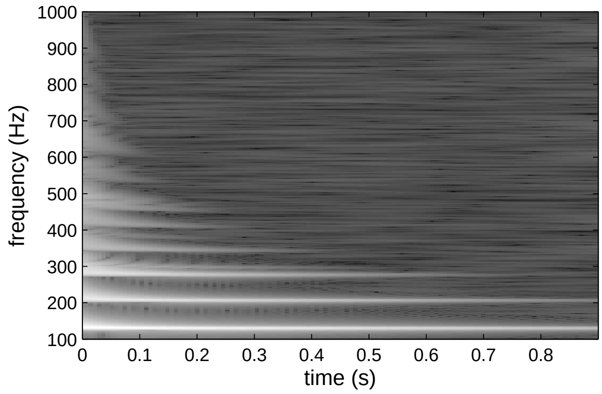
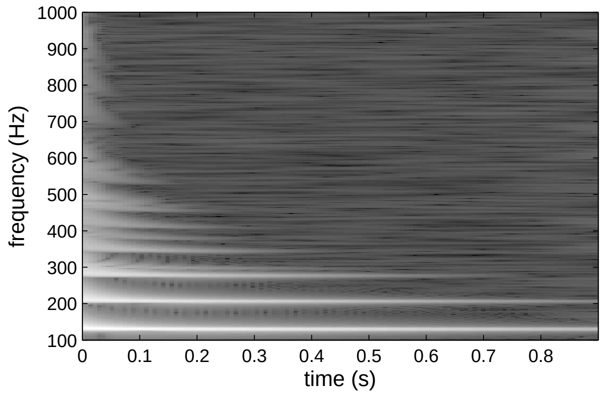
[2] www.dei.unipd.it/~avanzini/demos/dafx10



(a)



(b)



(c)

Figure 3: *Spectrograms of synthetic membrane sounds; (a) complete model using nonlinear tension T_{NL} of Eq. (5); (b) approximated model using quasistatic tension T_{qs} estimated from energy according to Eq. (20); (c) efficient model computing T_{qs} with a downsampling factor of* 32.

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