Quantum Noise of Actively Mode-Locked Lasers With Dispersion and Amplitude/Phase Modulation

Farhan Rana, Rajeev J. Ram, and Hermann A. Haus

***Abstract*—A quantum theory for the noise of optical pulses in actively mode-locked lasers is presented. In the presence of phase modulation and/or group velocity dispersion, the linear operator that governs the time evolution of the pulse fluctuations inside the laser cavity is not Hermitian (or normal) and the eigenmodes of this operator are not orthogonal. As a result, the eigenmodes have excess noise and the noise in different eigenmodes is highly correlated. We construct quantum operators for the pulse photon number, phase, timing, and frequency fluctuations. The nonorthogonality of the eigenmodes results in excess noise in the pulse photon number, phase, timing, and frequency. The excess noise depends on the frequency chirp of the pulse and is present only at low frequencies in the spectral densities of the pulse noise operators.**

***Index Terms*—Laser noise, optical pulses, quantum optics, ultrafast optics.**

I. Introduction

The noise of an optical pulse in a mode-locked laser can be determined by a perturbative expansion in terms of the eigenmodes of the linear operator that governs the slow time evolution of the pulse fluctuations [1]–[3]. In most mode-locked lasers,this operator is not Hermitian (i.e., not self-adjoint) or even normal (i.e., does not commute with its adjoint). This operator can be non-Hermitian in the presence of a number of different factors, such as group velocity dispersion, active phase modulation, dynamic gain or loss saturation, dynamic self-phase modulation, or detuning of the cavity round-rip frequency from the active modulation frequency. The eigenmodes of a nonnormal operator are not mutually orthogonal [4]. It is well known that the nonorthogonality of the eigenmodes significantly affects the noise in non-Hermitian (and nonnormal) optical systems [5]–[8]. The increased sensitivity to noise in phase-modulated lasers below threshold was pointed out in [9] and in detuned mode-locked lasers in [10]. In this paper, we present a quantum theory for the noise of optical pulses in mode-locked lasers in the presence of active phase/amplitude modulation and/or group velocity dispersion. Since the eigenmodes in this case are not orthogonal, the noise in eigenmodes is found by projections using the eigenmodes of the adjoint operator. As a result of the nonorthogonality of the eigenmodes, each eigenmode has large excess noise and the noise in different eigenmodes is highly correlated. We construct quantum mechanical operators for the pulse photon number, phase, timing, and frequency noise and show that these operators have noise contributions from eigenmodes of all order. The nonorthogonality of the eigenmodes also results in excess noise in the pulse photon number, phase, timing, and frequency. This excess noise has the same origin as the excess noise described by the Petermann's K-factor in non-Hermitian optical systems [5]–[8], and its magnitude depends on the degree of nonorthogonality of the eigenmodes. In the presence of active phase modulation and/or dispersion, the magnitude of the frequency chirp of the steady-state pulse is a good measure of the degree of nonorthogonality of the eigenmodes, and the excess noise in the pulse can be related to the pulse chirp. The spectral densities of the pulse photon number, phase, timing, and frequency noise exhibit excess noise only at frequencies lower than the smallest (in magnitude) nonzero eigenvalue of the operator that governs the time evolution of the pulse fluctuations.

Previous works on the noise in actively mode-locked lasers have either not taken into account group velocity dispersion and/or active phase modulation [2], [11], [12] or ignored the resulting non-Hermiticity [13]. The work presented in this paper is useful for understanding the noise of chirped optical pulses in mode-locked lasers and is especially relevant to semiconductor mode-locked lasers. Optical pulses in semiconductor mode-locked lasers can be highly chirped because of the large material dispersion and because phase modulation accompanies active amplitude modulation as a result of the carrier-density-dependent refractive index in semiconductors [14].

II. Theoretical Model

A. Master Equation for Actively Mode-Locked Lasers

We start from the time-domain perturbation theory for optical pulses developed in [1]–[3] and [15]. The model discussed in this paper is linear. Nonlinear effects, such as dynamic gain saturation and self-phase modulation, have not been included for simplicity. The quantum field operator for the optical pulse inside the laser cavity is $\hat{\phi}(t,T)\exp(-j\omega_o t)$, where $\hat{\phi}(t,T)$ describes the slowly varying envelope of the optical pulse and ω_o is the pulse center frequency. The additional time variable T describes the evolution of the pulse over time scales larger than the cavity round-trip time T_R. The operator $\hat{\phi}(t,T)$ is normalized such that $\langle\hat{\phi}^\dagger(t,T)\hat{\phi}(t,T)\rangle$ equals the photon number flux (units: number/s). The angled brackets $\langle\cdots\rangle$ stand for averaging

Manuscript received April 23, 2003; revised September 26, 2003. This work was supported by DARPA-PACT.

F. Rana is with the Cornell University, Ithaca, NY 14853 USA and also with Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

R. J. Ram is with the Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

H. A. Haus, deceased, was with the Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

Digital Object Identifier 10.1109/JQE.2003.820831

with respect to the quantum mechanical density operator that describes the steady state of the optical pulse. $\hat{\phi}(t,T)$ obeys the quantum mechanical commutation relation [3]

$$[\hat{\phi}(t,T), \hat{\phi}^{\dagger}(t',T)] = \delta(t-t'). \tag{1}$$

The Heisenberg–Langevin equation (known also as the master equation in the mode-locking literature) that describes the slow time evolution of the optical pulse inside the laser cavity is [1]–[3]

$$\frac{d\hat{\phi}(t,T)}{dT} = j\frac{\nu}{T_R}\hat{\phi}(t,T) + \left(g - \frac{1}{\tau_l}\right)\hat{\phi}(t,T) + O(t)\hat{\phi}(t,T) + \hat{F}_g(t,T) + \hat{F}_v(t,T) \tag{2}$$

where the operator $O(t)$ is

$$O(t) = (B - jD)\frac{1}{2}\frac{\partial^2}{\partial t^2} + (a_M + jp_M)[\cos(\Omega_R t) - 1] \tag{3}$$

where g is the pulse (amplitude) gain (units: 1/s), τ_l describes the photon loss from the laser cavity, B describes the effect of the finite filter bandwidth (units: s), D is the cavity group velocity dispersion (units: s). a_M and p_M are the strengths (units: 1/s) of amplitude and phase modulation, respectively, Ω_R is the frequency of the active modulation and it is assumed to be equal to $2\pi/T_R$, ν is a phase shift accumulated by the pulse in one round trip, and $\hat{F}_g(t,T)$ and $\hat{F}_v(t,T)$ are Langevin noise operators that describe the noise associated with gain and photon loss (or vacuum fluctuations), respectively, and have the following correlation functions:

$$\langle \hat{F}_g^{\dagger}(t,T)\hat{F}_g(t',T')\rangle = 2gn_{\text{sp}}\delta(t-t')\delta(T-T') \tag{4}$$

$$\langle \hat{F}_g(t,T)\hat{F}_g^{\dagger}(t',T')\rangle = 2g(n_{\text{sp}}-1)\delta(t-t')\delta(T-T') \tag{5}$$

$$\langle \hat{F}_v^{\dagger}(t,T)\hat{F}_v(t',T')\rangle = n_{\text{th}}\left[\frac{2}{\tau_l} - O(t) - O^{\dagger}(t)\right]\delta(t-t')\delta(T-T') \tag{6}$$

$$\langle \hat{F}_v(t,T)\hat{F}_v^{\dagger}(t',T')\rangle = (n_{\text{th}}+1)\left[\frac{2}{\tau_l} - O(t) - O^{\dagger}(t)\right]\delta(t-t')\delta(T-T') \tag{7}$$

where n_{sp} is the spontaneous emission factor that takes into account incomplete inversion of the gain medium [20], n_{th} is the thermal occupation number for photons at frequency ω_o and is close to zero since the photon energy $\hbar\omega_o$ is usually much greater than the thermal energy at room temperature. The noise operators obey the quantum mechanical commutation relations

$$[\hat{F}_g(t,T), \hat{F}_g^{\dagger}(t',T')] = -2g\delta(t-t')\delta(T-T') \tag{8}$$

$$[\hat{F}_v(t,T), \hat{F}_v^{\dagger}(t',T')] = \left[\frac{2}{\tau_l} - O(t) - O^{\dagger}(t)\right]\delta(t-t')\delta(T-T'). \tag{9}$$

Carrier number fluctuations in the gain medium will be included in the master equation later in this paper. It should be noted that the master equation (2) describes the pulse only over time scales longer than the cavity round-trip time, and the master equation is valid provided the pulse does not change significantly as it travels in the laser cavity.

B. Steady-State Solution

The steady-state solution is much simpler if the term $[\cos(\Omega_R(t) - 1]$ in (3) is approximated as $-\Omega_R^2 t^2/2$ [1]. The eigenfunctions $A_k(t)$ of the operator $O(t)$ are then complex Hermite–Gaussians

$$A_k(t) \propto H_k\left(\frac{t}{\tau}\sqrt{1+j\beta}\right)\exp\left[-\frac{t^2}{2\tau^2}(1+j\beta)\right] \tag{10}$$

where $H_k(\cdot)$ is the kth Hermite polynomial. The corresponding complex eigenvalues are λ_k. λ_k equals $(2k+1)\lambda_0$, where

$$\lambda_0 = -\frac{1}{2}\sqrt{(a_M + jp_M)(B - jD)\Omega_R^2}. \tag{11}$$

The master equation (2) is valid only if $|\lambda_0| \ll 1/T_R$. In most mode-locked semiconductor and fiber lasers, $|\lambda_0|$ is usually two or three orders of magnitude smaller than $1/T_R$ [15], [17]–[19]. The eigenfunctions $A_k(t)$ are normalized such that $\int dt\,|A_k(t)|^2 = 1$. The steady-state pulse is given by the eigenfunction $A_0(t)$ of the smallest (in magnitude) eigenvalue λ_0, where

$$A_0(t) = \frac{1}{(\sqrt{\pi}\tau)^{1/2}}\exp\left[-\frac{t^2}{2\tau^2}(1+j\beta)\right]. \tag{12}$$

The pulse chirp parameter β and the pulse width τ are given by the relations

$$\tan^{-1}(\beta) = \frac{1}{2}[\tan^{-1}(p_M/a_M) + \tan^{-1}(D/B)] \tag{13}$$

$$\tau^2 = \sqrt{1+\beta^2}\sqrt{\frac{1}{\Omega_R^2}\left(\frac{B^2+D^2}{a_M^2+p_M^2}\right)^{1/2}}. \tag{14}$$

The steady-state average value of the photon flux operator $\hat{\phi}^{+}(t,T)\hat{\phi}(t,T)$ is assumed to be

$$\langle\hat{\phi}^{+}(t,T)\hat{\phi}(t,T)\rangle = |\sqrt{n_o}A_0(t)|^2 \tag{15}$$

where n_o is the number of photons in the steady-state pulse. For steady-state pulse operation, the real and imaginary parts of the eigenvalue λ_0 satisfy

$$g = \frac{1}{\tau_l} - \text{Re}(\lambda_0) \tag{16}$$

$$\text{Im}(\lambda_0) + \frac{\nu}{T_R} = 0. \tag{17}$$

The total photon loss rate in the cavity is τ_p, given as

$$\frac{1}{\tau_p} = \frac{1}{\tau_l} - \text{Re}(\lambda_0). \tag{18}$$

Above laser threshold, $g = 1/\tau_p$. The gain g is assumed to be a decreasing function of the number of photons n_o in the optical pulse as a result of gain saturation. The details of the relationship between g and n_o are not important for the purposes of this

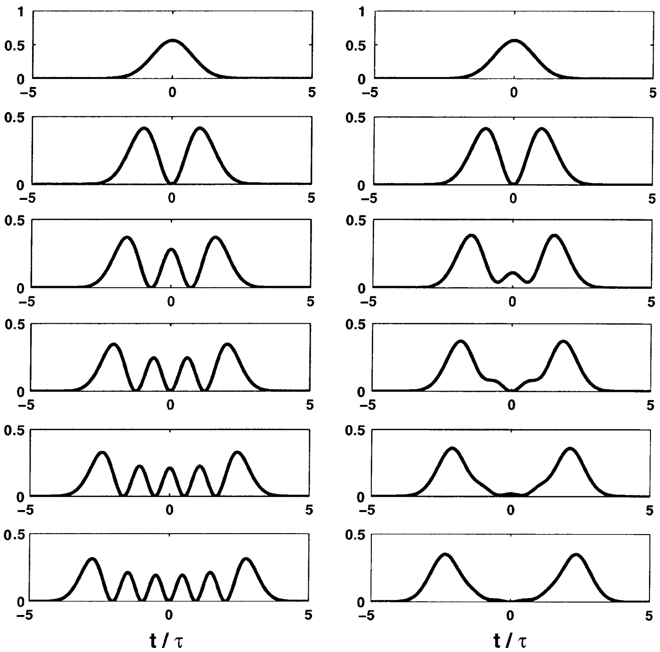

Fig. 1. Squared magnitudes of the first six eigenfunctions $A_k(t)$ are shown. β is 0 for the plots in the left column and 1 for the plots in the right column.

paper. However, it should be noted that (16) fixes the number of photons in the steady-state pulse. The interested reader is referred to [2] for details.

C. Nonorthogonal Eigenfunctions

The operator $O(t)$, in the presence of dispersion and/or phase modulation, is not Hermitian (or not self-adjoint) and not normal (i.e., it does not commute with its adjoint). Consequently, the eigenfunctions $A_k(t)$ are not orthogonal [4], i.e., $\int dt\, A_q^*(t)A_k(t) \neq \delta_{qk}$. However, $A_k(t)$ are orthogonal to the eigenfunctions of the adjoint operator $O^\dagger(t)$. The eigenfunctions of the operator $O^\dagger(t)$ are complex conjugates of the corresponding eigenfunctions of the operator $O(t)$ as

$$O^\dagger(t)A_k^*(t) = \lambda_k^* A_k^*(t). \tag{19}$$

The orthogonality relation is then $\int dt\, A_q(t)A_k(t) \propto \delta_{qk}$. We define the cross-product matrix M_{kq} as

$$M_{qk} = M_{kq}^* = \int dt\, A_q^*(t)A_k(t). \tag{20}$$

In Appendix I, it is shown that M_{kq} depends only on the magnitude of the pulse chirp β. When $\beta = 0$, the eigenfunctions are orthogonal and $M_{kq} = \delta_{kq}$. The degree of nonorthogonality of the eigenfunctions increases with the increase in the magnitude of the pulse chirp. The first six eigenfunctions for values of β equal to 0 and 1 are shown in Fig. 1. When $\beta \neq 0$, M_{kq} is nonzero if $A_k(t)$ and $A_q(t)$ have the same parity (i.e., if k and q are both even or both odd). Properties of the complex Hermite–Gaussian functions and expressions for the elements of the matrix M_{kq} are given in Appendix I. The eigenfunctions $A_k(t)$ form a complete basis set and satisfy the completeness relation

$$\sum_{k=0}^{\infty} \frac{A_k(t)A_k(t')}{\int dt\, A_k^2(t)} = \delta(t-t'). \tag{21}$$

III. Solution in the Presence of Noise

A. Eigenfunction Expansion

In the presence of noise the field operator $\hat{\phi}(t,T)$ can be written as a sum of a classical field, which describes the steady-state pulse, and a quantum operator that describes the noise and also preserves the field commutation relations [3], [16]

$$\hat{\phi}(t,T) = \sqrt{n_o}A_0(t) + \hat{\psi}(t,T) \tag{22}$$

where

$$\langle \hat{\psi}(t,T)\rangle = 0$$
$$[\hat{\psi}(t,T), \hat{\psi}^\dagger(t',T)] = \delta(t-t'). \tag{23}$$

The operator $\hat{\psi}(t,T)$ can be expanded in terms of the eigenfunctions of the operator $O(t)$ as

$$\psi(t,T) = \sqrt{n_o}\sum_{k=0}^{\infty} \hat{c}_k(T)A_k(t) \tag{24}$$

where $\hat{c}_k(T)$ is a quantum mechanical annihilation operator. The operators $\hat{c}_k(T)$ obey the commutation relations

$$[\hat{c}_k(T), \hat{c}_q^\dagger(T)] = \frac{M_{qk}}{n_o \int dt\, A_q^{*2}(t) \int dt\, A_k^2(t)}. \tag{25}$$

If the eigenfunctions were orthogonal, then the terms inside the square bracket in (25) would equal δ_{kq}. The operator $\hat{c}_k(T)$ annihilates a photon in the mode $A_k(t)$. However, the operator $\hat{c}_k^\dagger(T)$ creates a photon in the mode $A_k^*(t)$ and not in the mode $A_k(t)$. This can be shown as follows. The temporal wave-function of the photon created by the operator $\hat{c}_k^\dagger(T)$ can be obtained by looking at the probability amplitude when the state is destroyed by $\hat{\psi}(t,T)$ as follows:

$$\hat{\psi}(t,T)\hat{c}_k^\dagger(T)|0\rangle = [\hat{\psi}(t,T), \hat{c}_k^\dagger(T)]|0\rangle = \frac{A_k^*(t)}{\sqrt{n_0}\int dt\, A_k^{*2}(t)}|0\rangle. \tag{26}$$

B. Noise Dynamics and Excess Noise

The dynamical equation for the operator $\hat{c}_k(T)$ can be found by substituting the eigenfunction expansion in (24) in the master equation (2), and using the eigenfunction $A_k^*(t)$ of the adjoint operator $O^\dagger$ to project out the equation for $\hat{c}_k(T)$ as

$$\frac{d\hat{c}_k(T)}{dT} = (\lambda_k - \lambda_0)\hat{c}_k(T) + \frac{\int dt[\hat{F}_g(t,T) + \hat{F}_v(t,T)]A_k(t)}{\sqrt{n_o}\int dt\, A_k^2(t)}. \tag{27}$$

The commutation relations given in (8) and (9) for the noise sources $\hat{F}_g(t,T)$ and $\hat{F}_v(t,T)$ preserve the commutation relations for the operators $\hat{c}_k(T)$ during time evolution. The equation for the operator $\hat{c}_0(T)$ in (27) is not damped. Carrier number fluctuations (or gain fluctuations) must be included in the model to damp the fluctuations in $\hat{c}_0(T)$, and this will be done in Section V. For $k \neq 0$, (27) can be integrated directly to yield the expectation values for the operators $\hat{c}_k(T)$, and, for $k, q \neq 0$ we obtain

$$\begin{aligned}\langle \hat{c}_k(T)\rangle &= \langle \hat{c}_k^\dagger(T)\rangle = \langle \hat{c}_k(T)\hat{c}_q(T)\rangle \\ &= \langle \hat{c}_k^\dagger(T)\hat{c}_q^\dagger(T)\rangle = 0 \end{aligned} \tag{28}$$

$$\begin{aligned}\langle \hat{c}_k^\dagger(T)\hat{c}_q(T)\rangle &= \frac{\frac{1}{\tau_p}(n_{\rm th}+n_{\rm sp}) - (k\lambda_0^* + q\lambda_0)n_{\rm th}}{-(k\lambda_0^* + q\lambda_0)} \\ &\quad \times \left[\frac{M_{kq}}{n_o \int dt\, A_k^{*2}(t) \int dt\, A_q^2(t)}\right] \end{aligned} \tag{29}$$

$$\begin{aligned}\langle \hat{c}_q(T)\hat{c}_k^\dagger(T)\rangle &= \frac{\frac{1}{\tau_p}(n_{\rm th}+n_{\rm sp}) - (k\lambda_0^* + q\lambda_0)(n_{\rm th}+1)}{-(k\lambda_0^* + q\lambda_0)} \\ &\quad \times \left[\frac{M_{kq}}{n_o \int dt\, A_k^{*2}(t) \int dt\, A_q^2(t)}\right]. \end{aligned} \tag{30}$$

The term $(k\lambda_0^* + q\lambda_0)$ in the numerator in (29) and (30) can be ignored if $|\lambda_0| \ll 1/\tau_p$. Since τ_p is usually of the order of T_R in most mode-locked lasers, the condition $|\lambda_0| \ll 1/\tau_p$ is also satisfied if $|\lambda_0| \ll 1/T_R$. Since the master equation (2) is valid only if $|\lambda_0| \ll 1/T_R$, these terms will be ignored in this paper.

The expectation values $\langle \hat{c}_k(T)\hat{c}_k^\dagger(T)\rangle$ and $\langle \hat{c}_k^\dagger(T)\hat{c}_k(T)\rangle$ are proportional to $1/|\int dt\, A_k^2(t)|^2$ which is always greater than or equal to unity (this follows from the Schwartz's inequality). $1/|\int dt\, A_k^2(t)|^2$ is the excess noise factor. The excess noise would have been absent if the eigenfunctions were orthogonal. The excess noise factor is similar to the Petermann's K-factor which describes the excess noise in optical amplifiers and oscillators with nonorthogonal optical modes [5]–[8]. The excess noise factors depend on the magnitude of the pulse chirp (see Appendix I) as follows:

$$\frac{1}{\left|\int dt\, A_k^2(t)\right|^2} = |\sqrt{1+j\beta}P_k(\sqrt{1+\beta^2})|^2 \tag{31}$$

where $P_k(\cdot)$ is the kth Legendre polynomial, $P_0(x) = 1$, and $P_1(x) = x$. The excess noise depends on the degree of nonorthogonality of the eigenfunctions (or on the degree of nonnormality of the operator $O(t)$). The larger the magnitude of the pulse chirp, the more nonorthogonal the eigenfunctions, and the larger the excess noise. Also, (29) and (30) show that as a result of the nonorthogonality of the eigenfunctions the noise in different eigenfunctions is correlated.

C. Pulse Fluctuation Operators

In many cases the quantities of interest are usually the pulse photon number, phase, timing, and frequency fluctuations. The eigenfunction expansion for the operator $\hat{\psi}(t,T)$ in (24) can be written as [2], [3], [16],

$$\begin{aligned}\hat{\psi}(t,T) &= \sqrt{n_o}\left[\left(\frac{\delta\hat{n}(T)}{2n_o} + j\delta\hat{\theta}(T)\right)A_o(t) \right.\\ &\quad \left. - \delta\hat{t}(T)\frac{\partial A_o(t)}{\partial t} - j\delta\hat{\omega}(T)tA_o(t)\right] \\ &\quad + \sqrt{n_o}\sum_{k=2}^{\infty}\hat{c}_k(T)A_k(t) \end{aligned} \tag{32}$$

where the contribution from the third and higher order eigenfunctions has been separated. The expansion in (32) has been widely used in the literature [2], [3], [16], and the operators $\delta\hat{n}(T), \delta\hat{\theta}(T), \delta\hat{t}(T)$, and $\delta\hat{\omega}(T)$ have been identified with the pulse photon number, phase, timing, and frequency fluctuations, respectively. The drawback with the expansion in (32) is that when the eigenfunctions are not orthogonal then the perturbations given by the higher order eigenfunctions are not orthogonal to the perturbations given by the first two eigenfunctions. Consequently, the operators for the pulse photon number, phase, timing, and frequency fluctuations as defined in (32) are not valid and do not correspond to the quantities measured in experiments [17]–[19]. Below, operators are constructed that describe the pulse photon number, phase, timing, and frequency fluctuations when the eigenfunctions of the operator $O(t)$ are not orthogonal. In Section VII, it is shown that the operators defined below correspond to the quantities measured in experiments.

The total number of photons in the pulse at time T is given by the operator

$$\int dt\, \hat{\phi}^\dagger(t,T)\hat{\phi}(t,T). \tag{33}$$

The operator $\Delta\hat{n}(T)$ for the fluctuations in the pulse photon number can be obtained by using (22) in (33) and keeping only the term's first order in $\hat{\psi}(t,T)$ as

$$\Delta\hat{n}(T) = \sqrt{n_o}\int dt\, A_0^*(t)\hat{\psi}(t,T) + \text{h.c.} \tag{34}$$

The operator $\Delta\hat{\theta}(T)$ for the pulse phase fluctuations, which is also conjugate to $\Delta\hat{n}(T)$, is

$$\Delta\hat{\theta}(T) = \frac{1}{\sqrt{n_o}}\int dt\, \frac{1}{2j}A_0^*(t)\hat{\psi}(t,T) + \text{h.c.} \tag{35}$$

The correct commutation relation between $\Delta\hat{n}(T)$ and $\Delta\hat{\theta}(T)$ follow from the commutation relation between $\hat{\psi}(t,T)$ and $\hat{\psi}^\dagger(t',T)$,:

$$[\Delta\hat{n}(T), \Delta\hat{\theta}(T)] = j. \tag{36}$$

The operator for the pulse position in time is

$$\int dt\, \hat{\phi}^\dagger(t,T)t\hat{\phi}(t,T). \tag{37}$$

It follows that the operator $\Delta\hat{t}(T)$ for the fluctuations in the pulse timing is

$$\Delta\hat{t}(T) = \frac{1}{\sqrt{n_o}}\int dt\, tA_0^*(t)\hat{\psi}(t,T) + \text{h.c.} \tag{38}$$

The operator $\Delta\hat{\omega}(T)$ for the pulse frequency fluctuations, which is also conjugate to $\Delta\hat{t}(T)$, is,

$$\Delta\hat{\omega}(T) = \frac{1}{\sqrt{n_o}}\int dt\, \frac{1}{j}\frac{\partial A_0^*(t)}{\partial t}\hat{\psi}(t,T) + \text{h.c.} \tag{39}$$

The commutation relation between $\Delta\hat{\omega}(T)$ and $\Delta\hat{t}(T)$ is

$$[\Delta\hat{\omega}(T), \Delta\hat{t}(T)] = \frac{j}{n_o}. \tag{40}$$

Using the eigenfunction expansion for $\hat{\psi}(t,T)$, the expressions for the pulse fluctuation operators become

$$\Delta\hat{n}(T) = n_o \sum_{k=0}^{\infty} M_{0k}\hat{c}_k(T) + \text{h.c.} \tag{41}$$

$$\Delta\hat{\theta}(T) = \frac{1}{2j} \sum_{k=0}^{\infty} M_{0k}\hat{c}_k(T) + \text{h.c.} \tag{42}$$

$$\Delta\hat{t}(T) = \frac{\tau}{\sqrt{2}} \frac{(1+j\beta)^{1/2}}{(1+\beta^2)^{1/4}} \sum_{k=0}^{\infty} M_{1k}\hat{c}_k(T) + \text{h.c.} \tag{43}$$

$$\Delta\hat{\omega}(T) = \frac{j}{\sqrt{2}\tau} \frac{(1+\beta^2)^{3/4}}{(1+j\beta)^{1/2}} \sum_{k=0}^{\infty} M_{1k}\hat{c}_k(T) + \text{h.c.} \tag{44}$$

Equations (41)–(44) show that the noise in all the higher order eigenfunctions contribute to the pulse photon number, phase, timing, and frequency fluctuations. If the expansions in (41)–(44) are limited to only the first two eigenfunctions, then the expressions for $\delta\hat{n}(T), \delta\hat{\theta}(T), \delta\hat{t}(T)$, and $\delta\hat{\omega}(T)$, respectively, are obtained. In the sections that follow, the noise in pulse photon number, phase, timing, and frequency are discussed in detail.

IV. Pulse Timing and Frequency Noise

The operators $\Delta\hat{t}(T)$ and $\Delta\hat{\omega}(T)$ for the pulse timing and frequency noise, respectively, contain noise contributions from all the odd-numbered eigenfunctions. The excess noise and the correlation in the noise in different eigenfunctions significantly affect the pulse timing and frequency noise. The operators $\delta\hat{t}(T)$ and $\delta\hat{\omega}(T)$ contain noise contribution from only the second eigenfunction and exhibit the excess noise in the second eigenfunction. The mean square value of $\delta\hat{t}(T)$ can be obtained using (29) and (30) to yield

$$\begin{aligned}\langle\delta\hat{t}^2(T)\rangle &= \frac{\tau^2}{2}[\langle\hat{c}_1(T)\hat{c}_1^\dagger(T)\rangle + \langle\hat{c}_1^\dagger(T)\hat{c}_1(T)\rangle] \\ &\approx \frac{\tau^2}{2n_o}\left[\frac{2n_{\text{sp}}}{(-\lambda_0-\lambda_0^*)\tau_p}\right]\frac{1}{|\int dt\, A_1^2(t)|^2} \\ &= \left(\frac{n_{\text{sp}}}{n_o}\right)\frac{(1+\beta^2)^{5/2}}{\tau_p\Omega_R^2(a_M+\beta p_M)}.\end{aligned} \tag{45}$$

The mean square value of $\delta\hat{\omega}(T)$ can be obtained from the mean square value of $\delta\hat{t}(T)$ as

$$\begin{aligned}\langle\delta\hat{\omega}^2(T)\rangle &= \frac{\langle\delta\hat{t}^2(T)\rangle}{\tau^4}(1+\beta^2) \\ &\approx \left(\frac{n_{\text{sp}}}{n_o}\right)\frac{(1+\beta^2)^{7/2}}{\tau_p\Omega_R^2\tau^4(a_M+\beta p_M)}.\end{aligned} \tag{46}$$

The factor $(1+\beta^2)^{3/2}(=1/|\int dt\, A_1^2(t)|^2)$ in (45) and (46) is the excess noise.

The mean square value of $\Delta\hat{t}(T)$ can be calculated using (29), (30), and (43) to yield

$$\begin{aligned}\langle\Delta\hat{t}^2(T)\rangle &= \frac{\tau^2}{2}\sum_{k,q=0}^{\infty} M_{1k}M_{q1}[\langle\hat{c}_k(T)\hat{c}_q^\dagger(T)\rangle \\ &\quad + \langle\hat{c}_q^\dagger(T)\hat{c}_k(T)\rangle] \\ &\approx \left(\frac{n_{\text{sp}}}{n_o}\right)\frac{\tau^2}{\tau_p}\sum_{k,q=0}^{\infty}\frac{M_{1k}M_{qk}M_{q1}}{(-k\lambda_0-q\lambda_0^*)} \\ &\quad \times \frac{1}{\int dt\, A_q^{*2}(t)\int dt\, A_k^2(t)}.\end{aligned} \tag{47}$$

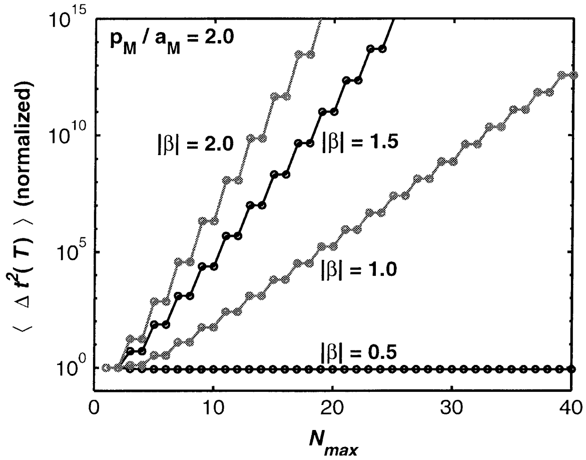

Fig. 2. Mean square timing noise $\langle\Delta\hat{t}^2(T)\rangle$ calculated using (47), normalized to the value of $\langle\delta\hat{t}^2(T)\rangle$ in (45), is plotted as a function of the number $N_{\max}$ of eigenfunctions used in the expansion. The perturbative expansion diverges exponentially when the pulse chirp $|\beta|$ becomes larger than $1/\sqrt{3} \approx 0.577$. The steps appear because only the odd-numbered eigenfunctions contribute to the timing noise.

The first nonzero term ($k=q=1$) of the series in (47) equals $\langle\delta\hat{t}^2(T)\rangle$ given in (45). Fig. 2 shows $\langle\Delta\hat{t}^2(T)\rangle$ calculated using (47) as a function of the number of eigenfunctions $N_{\max}$ used in the expansion [i.e., when only terms with $k,q < N_{\max}$ are included in the summation in (47)]. When the magnitude of the pulse chirp β is larger than $1/\sqrt{3} \approx 0.577$, the series in (47) diverges exponentially. The perturbative expansion for the pulse frequency noise $\Delta\hat{\omega}(T)$ also diverges in the same way. The divergence of the perturbative expansion is analyzed in detail in Appendix II, and it is shown that, in general, a series of the following form:

$$\sum_{k,q=0}^{\infty} F_{qk}\frac{M_{pk}M_{qk}M_{qp}}{\int dt\, A_q^{*2}(t)\int dt\, A_k^2(t)} \tag{48}$$

(where F_{qk} $(=F_{kq}^*)$ decays only algebraically as k,q become large) does not converge when $|\beta| > 1/\sqrt{3}$. The divergence occurs because the eigenfunctions are highly nonorthogonal when $|\beta| > 1/\sqrt{3}$ and are not a suitable basis to describe the pulse noise. This divergence is related to the divergence observed in [21] in the general context of functional expansions using complex Hermite–Gaussians. The divergence can be removed by an appropriate change of the basis. In Appendix II, the minimum-error expansion of [21] is used to establish the following result:

$$\begin{aligned}&\sum_{k,q=0}^{\infty} F_{qk}\frac{M_{pk}M_{qk}M_{qp}}{\int dt\, A_q^{*2}(t)\int dt\, A_k^2(t)} \\ &= \lim_{N_{\max}\to\infty}\left|\int dt\, A_p^2(t)\right|^2\sum_{k,q=0}^{N_{\max}-1} F_{qk}M_{kp}^{-1}M_{qk}M_{pq}^{-1}.\end{aligned} \tag{49}$$

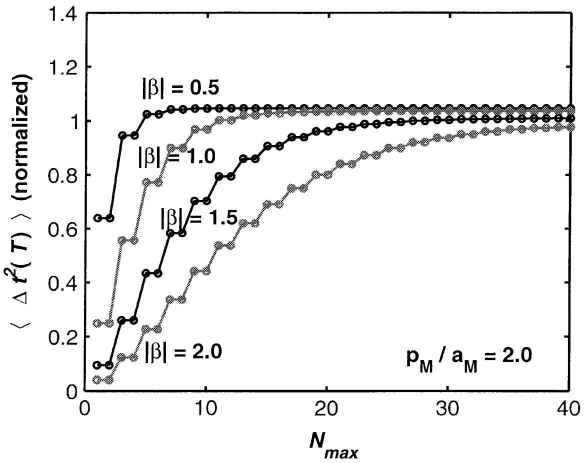

Fig. 3. Mean square timing noise $\langle \Delta \hat{t}^2(T) \rangle$, calculated using (50) and normalized to the expression given in (51), is plotted as a function of the number $N_{\max}$ of eigenfunctions used in the perturbative expansion. The series in (50) converges for all values of the pulse chirp.

The series on the right-hand side in the above equation converges for all values of the pulse chirp β as $N_{\max}$ becomes large. The relation in (49) will be used throughout this paper. Using (47) and (49), we obtain

$$\langle \Delta \hat{t}^2(T) \rangle \approx \left(\frac{n_{\rm sp}}{n_o}\right) \frac{\tau^2}{\tau_p(1+\beta^2)^{3/2}} \times \sum_{k,q=0}^{N_{\max}-1} \frac{M_{k1}^{-1} M_{qk} M_{1q}^{-1}}{(-k\lambda_0 - q\lambda_0^*)}. \quad (50)$$

The series above converges to a value which may be approximated (with less than 5% error) by the expression

$$\langle \Delta \hat{t}^2(T) \rangle \sim \left(\frac{n_{\rm sp}}{n_o}\right) \frac{(1+\beta^2)^{3/2}}{\tau_p \Omega_R^2 (a_M + \beta p_M)}. \quad (51)$$

The convergence of the expansion in (50) is shown in Fig. 3, where $\langle \Delta \hat{t}^2(T) \rangle$, normalized to the approximate expression given in (51), is plotted as a function of $N_{\max}$ for $p_M/a_M = 3.0$. Comparing (45) and (51), it is seen that the dominant effect of the noise contribution from the higher order eigenfunctions on the mean square value of the timing noise $\Delta \hat{t}(T)$ is the reduction of the excess noise from $(1+\beta^2)^{3/2}$ to $(1+\beta^2)^{1/2}$. The reduction in the noise is due to the correlations in the noise in different eigenfunctions.

It needs to be emphasized here that the excess noise in the pulse would not have disappeared if an orthogonal basis were used to expand the operator $\hat{\psi}(t,T)$ in (24) instead of the eigenfunction basis. The excess noise is a result of the nonnormality of the operator $O(t)$ governing the time evolution of the pulse noise and is independent of the basis set used. If a basis consisting of orthogonal functions is used, then the noise in different functions would be coupled since these functions would not be the eigenfunctions of the operator $O(t)$, and the excess noise would then appear as a result of these couplings. The description of noise dynamics in terms of the eigenfunction basis is the simplest.

Pulse stability with respect to timing perturbations require that $(1+\beta p_M/a_M) > 0$. Using (13), it can be shown that this stability condition is always satisfied for all values of D/B and p_M/a_M. The expression in (51) can be written as

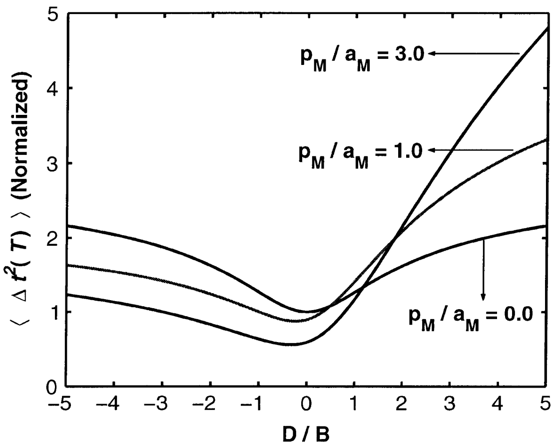

Fig. 4. Mean square timing noise $\langle \Delta \hat{t}^2(T) \rangle$, normalized to σ_a^2 (see (53), is plotted as a function of the ratio D/B for different values of the ratio p_M/a_M. The corresponding curves for negative values of p_M/a_M are reflections of the curves shown about the vertical axis.

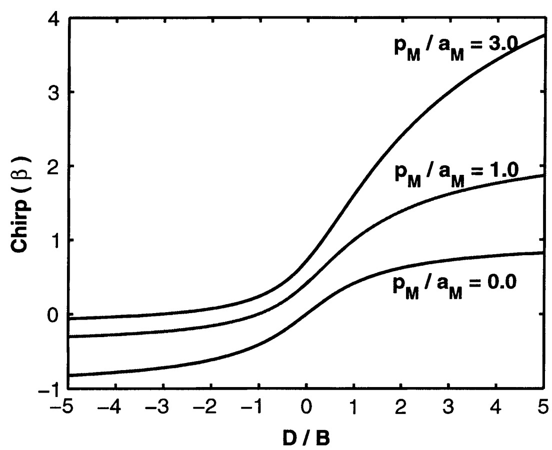

Fig. 5. Pulse chirp β is plotted as a function of the ratio D/B for different values of the ratio p_M/a_M. The corresponding curves for negative values of p_M/a_M are reflections of the curves shown about the vertical axis and horizontal axis.

$$\langle \Delta \hat{t}^2(T) \rangle \approx \sigma_a^2 \frac{(1+\beta^2)^{3/2}}{(1+\beta p_M/a_M)} \quad (52)$$

where σ_a^2 is given by the expression

$$\sigma_a^2 = \left(\frac{n_{\rm sp}}{n_o}\right) \frac{1}{\tau_p \Omega_R^2 a_M} \quad (53)$$

and is the mean square timing noise in the absence of group velocity dispersion and phase modulation. Fig. 4 shows $\langle \Delta \hat{t}^2(T) \rangle$, normalized to σ_a^2, plotted as a function of the ratio D/B for different values of the ratio p_M/a_M. The corresponding values of the pulse chirp are shown in Fig. 5.

The mean square timing noise is minimum when the dispersion is such that

$$4\beta \frac{p_M}{a_M} = \sqrt{9 + 8\left(\frac{p_M}{a_M}\right)^2} - 3. \quad (54)$$

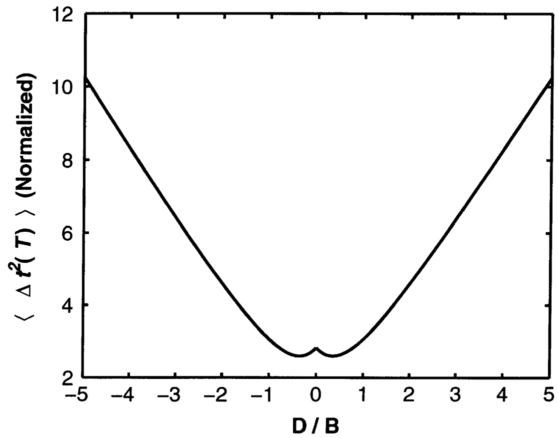

Fig. 6. Mean square timing noise $\langle \Delta \hat{t}^2(T) \rangle$, normalized to σ_p^2 [see (56)], is plotted as a function of the ratio D/B when pure phase modulation is used.

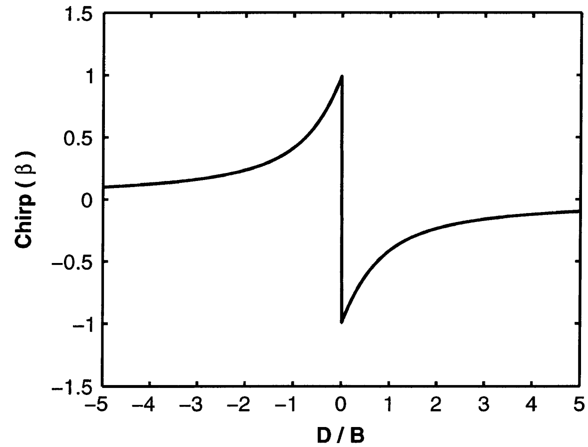

Fig. 7. Pulse chirp β is plotted as a function of the ratio D/B when pure phase modulation is used.

In the presence of phase modulation, the minimum timing noise always occurs for a nonzero value of the pulse chirp, as shown in Figs. 4 and 5.

When pure phase modulation is used (i.e., when $a_M = 0$), the pulse can be at the crest or trough of the sinusoidal modulation signal where the phase modulation has opposite signs. Using (11) with (16), and assuming $p_M > 0$, it can be shown that when $D < 0$ $(D > 0)$ the threshold gain is lower if the pulse is at the crest (trough), and therefore only one solution will be stable. Pulse stability with respect to timing perturbations requires that $\beta p_M > 0$ which, using (13), can be shown to be satisfied for all values of D/B. For pure phase modulation, (51) can be written as

$$\langle \Delta \hat{t}^2(T) \rangle \approx \sigma_p^2 \frac{(1+\beta^2)^{3/2}}{\beta} \tag{55}$$

where σ_p^2 is

$$\sigma_p^2 = \left(\frac{n_{\rm sp}}{n_o}\right) \frac{1}{\tau_p \Omega_R^2 p_M}. \tag{56}$$

Fig. 6 shows $\langle \Delta \hat{t}^2(T) \rangle$, normalized to σ_p^2, plotted as a function of the ratio D/B. The corresponding values of the pulse chirp are shown in Fig. 7. Assuming a fixed value of p_M, the mean square timing noise is minimum when the group velocity dispersion is such that $\beta^2 = 1/2$ (and $\beta p_M > 0$).

The mean square value of the pulse frequency noise $\Delta\hat{\omega}(T)$ can be from the mean square value of the timing noise $\Delta\hat{t}(T)$ as follows:

$$\begin{aligned}\langle \Delta \hat{\omega}^2(T) \rangle &= \frac{\langle \hat{t}^2(T) \rangle}{\tau^4}(1+\beta^2) \\ &\sim \left(\frac{n_{\rm sp}}{n_o}\right) \frac{(1+\beta^2)^{5/2}}{\tau_p \Omega_R^2 \tau^4 (a_M + \beta p_M)}.\end{aligned} \tag{57}$$

As in the case of the pulse timing noise, the noise contribution from the higher order eigenfunctions reduces the excess noise in the mean square pulse frequency noise from $(1+\beta^2)^{3/2}$ to $(1+\beta^2)^{1/2}$.

A. Spectral Densities of the Timing and Frequency Noise

The pulse timing and frequency noise spectral densities can be determined by solving (27) in the frequency domain. The spectral density $S_{\delta t}(\Omega)$ of the noise operator $\delta\hat{t}(T)$ follows from (139) of Appendix V-A:

$$\begin{aligned}S_{\delta t}(\Omega) &= \frac{\tau^2}{4} \int [\langle \hat{c}_1(\Omega) \hat{c}_1^\dagger(\Omega') \rangle + \langle \hat{c}_1^\dagger(\Omega) \hat{c}_1(\Omega') \rangle] \frac{d\Omega'}{2\pi} \\ &\quad + (\Omega \to -\Omega) \qquad (58)\\ &\approx \left(\frac{n_{\rm sp}}{n_o}\right) \frac{2\tau^2}{\tau_p} \\ &\quad \times \frac{\Omega^2 + |2\lambda_0|^2}{(\Omega^2 - |2\lambda_0|^2)^2 + \Omega^2(-2\lambda_0 - 2\lambda_0^*)^2} \\ &\quad \times (1+\beta^2)^{3/2}. \qquad (59)\end{aligned}$$

As a result of the coupling between $\delta\hat{t}(T)$ and $\delta\hat{\omega}(T)$ in the presence of dispersion and/or active phase modulation [2], [3], the frequency dependence of the spectral density $S_{\delta t}(\Omega)$ is that of a second-order linear system with the damping constant equal to $-4\mathrm{Re}(\lambda_0)$ and the square of the resonant frequency equal to $|2\lambda_0|^2$. The factor $(1+\beta^2)^{3/2}$ in (59) is the excess noise factor. The spectral density $S_{\delta\omega}(\Omega)$ of the operator $\delta\hat{\omega}(T)$ equals $(1+\beta^2)S_{\delta t}(\Omega)/\tau^4$.

The spectral density $S_{\Delta t}(\Omega)$ of the timing noise $\Delta\hat{t}(T)$ can be found using (43) and (139). The resulting series does not converge and the relation in (49) can be used to obtain the following convergent expansion for $S_{\Delta t}(\Omega)$:

$$\begin{aligned}S_{\Delta t}(\Omega) &\approx \left(\frac{n_{\rm sp}}{n_o}\right) \frac{2\tau^2}{\tau_p (1+\beta^2)^{3/2}} \\ &\quad \times \sum_{k,q=0}^{N_{\max}-1} \frac{M_{k1}^{-1} M_{qk} M_{1q}^{-1} (\Omega^2 + kq|2\lambda_0|^2)}{(\Omega^2 - kq|2\lambda_0|^2)^2 + \Omega^2(-2k\lambda_0 - 2q\lambda_0^*)^2}.\end{aligned} \tag{60}$$

Fig. 8 shows the timing noise spectral densities $S_{\delta t}(\Omega)$ and $S_{\Delta t}(\Omega)$, both normalized to the value $S_{\delta t}(\Omega = 0)/(1+\beta)^{3/2}$, for $\beta = 2.0$ and $p_M/a_M = 3.0$. The excess noise contributes to $S_{\delta t}(\Omega)$, a constant factor $(1+\beta)^{3/2}$. Since the noise in different eigenfunctions is correlated, the inclusion of noise contributions from the higher order eigenfunctions in $S_{\Delta t}(\Omega)$ reduces

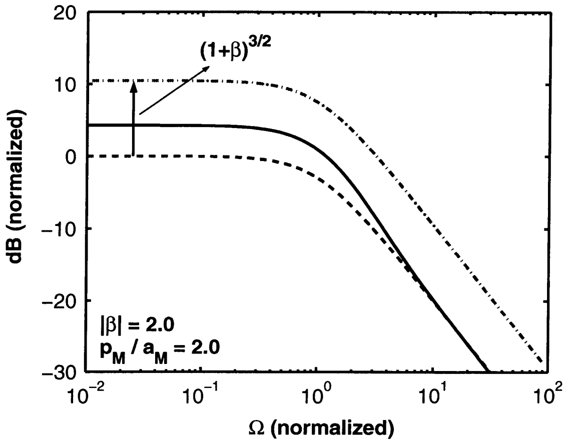

Fig. 8. Timing noise spectral densities $S_{\delta t}(\Omega)$ (dash–dotted line) and $S_{\Delta t}(\Omega)$ (solid line) are shown for $\beta = 2.0$ and $p_M/a_M = 2.0$. The dashed line shows $S_{\delta t}(\Omega)/(1+\beta^2)^{3/2}$. The spectral densities in the figure have been normalized w.r.t. the value $S_{\delta t}(\Omega = 0)/(1+\beta^2)^{3/2}$. The frequency Ω has been normalized w.r.t. the value $|2\lambda_0|$.

the low-frequency noise and partially compensates for the excess noise. At high frequencies, when $\Omega \gg |2\lambda_0|$, the series in (60) can be summed exactly and we obtain

$$S_{\Delta t}(\Omega \gg |2\lambda_0|) = \left(\frac{n_{\rm sp}}{n_o}\right)\frac{2\tau^2}{\tau_p \Omega^2} = \frac{S_{\delta t}(\Omega)}{(1+\beta)^{3/2}}. \tag{61}$$

The above equation shows that the excess noise in the pulse timing noise spectrum is absent at high frequencies. This is also shown in Fig. 8. The excess noise is therefore present only at frequencies lower than the smallest (in magnitude) nonzero eigenvalue of the operator $O(t)$. The spectral density $S_{\Delta\omega}(\Omega)$ of the frequency noise $\Delta\hat{\omega}(T)$ is simply $(1+\beta^2)S_{\Delta t}(\Omega)/\tau^4$. In the next section, it is shown that the excess noise in the pulse photon number and phase are also present only at low frequencies.

V. Photon Number and Phase Fluctuations

Carrier number fluctuations (or gain fluctuations) must be included in the model to determine the pulse photon number and phase fluctuations. Here we assume a semiconductor gain medium. Carrier number noise $\Delta\hat{N}(T)$ introduces an additional term

$$\frac{\partial g}{\partial N}(1-j\alpha)\Delta\hat{N}(T)\sqrt{n_o}A_0(t) \tag{62}$$

on the left-hand side of the master equation (2). The parameter α relates the change in the imaginary part of the gain to the change in the real part of the gain and models the refractive index fluctuations which accompany gain fluctuations [20]. Carrier number fluctuations affect only the dynamical equation of $\hat{c}_0(T)$ as

$$\frac{d\hat{c}_0(T)}{dT} = \frac{(1-j\alpha)}{2\tau_{\rm st} n_o}\Delta\hat{N}(T) + \frac{\int dt\,[\hat{F}_g(t,T)+\hat{F}_v(t,T)]A_0(t)}{\sqrt{n_o}\int dt\,A_0^2(t)} \tag{63}$$

where $\tau_{\rm st}$ is the carrier stimulated emission lifetime

$$\frac{1}{\tau_{\rm st}} = 2\frac{\partial g}{\partial N}n_o. \tag{64}$$

When the nonradiative recombination time $\tau_{\rm nr}$ of the carriers is much shorter than the pulse round-trip time T_R—a condition true for most semiconductor and fiber mode-locked lasers—the dynamical equation for the carrier number fluctuations is

$$\frac{d\Delta\hat{N}(T)}{dT} = -\left(\frac{1}{\tau_{\rm nr}}+\frac{1}{\tau_{\rm st}}\right)\Delta\hat{N}(T) - 2g\Delta\hat{n}(T) - \sqrt{n_o}\sum_{k=0}^{\infty}\left[M_{0k}\frac{\int dt\,F_g(t,T)A_k(t)}{\int dt\,A_k^2(t)} + \text{h.c.}\right] + \hat{F}_{\rm pump}(T) + \hat{F}_{\rm nr}(T). \tag{65}$$

The first term on the right-hand side of the above equation is the increase in the carrier recombination rate due to an increase in the carrier number. The second term is the increase in the carrier recombination rate due to an increase in the number of photons in the pulse. The third term describes the noise associated with spontaneous emission and vacuum fluctuations. $\hat{F}_{\rm nr}(T)$ models the noise in carrier nonradiative recombination, and $\hat{F}_{\rm pump}(T)$ models the noise in the pumping process. If the pumping process has shot noise, then

$$\langle \hat{F}_{\rm pump}(T)\hat{F}_{\rm pump}(T')\rangle = R_{\rm pump}\delta(T-T'). \tag{66}$$

$R_{\rm pump}$ is the average rate at which carriers are pumped into the upper level. The correlation function of $\hat{F}_{\rm nr}(T)$ is

$$\langle \hat{F}_{\rm nr}(T)\hat{F}_{\rm nr}(T')\rangle \approx R_{\rm nr}\delta(T-T'). \tag{67}$$

$R_{\rm nr}$ is the average carrier recombination rate. It can be shown that (63) and (65), together with (27), preserve the commutation relations for the operators $\hat{c}_k(T)$ given in (25).

A. Photon Number Fluctuations

The spectral density of the pulse photon number fluctuations can be found by solving (63) and (65) in the frequency domain. The modulation response $H(\Omega)$ of the laser is defined as [20]

$$H(\Omega) = \frac{\Omega_{\rm ro}^2}{\Omega_{\rm ro}^2 - \Omega^2 + j\Omega\gamma} \tag{68}$$

where the laser relaxation oscillation frequency $\Omega_{\rm ro}$ and the damping constant γ are

$$\Omega_{\rm ro}^2 = \frac{2}{\tau_{\rm st}\tau_p} \tag{69}$$

$$\gamma = \frac{1}{\tau_{\rm st}} + \frac{1}{\tau_{\rm nr}}. \tag{70}$$

Equation (65) for the carrier number fluctuations is valid only if $\Omega_{\rm ro}$ and γ are much smaller than $1/T_R$. Recall from the discussion in Section III-C that $\delta\hat{n}(T)$ is an approximation to the pulse photon number fluctuation operator $\Delta\hat{n}(T)$ and is obtained by restricting the eigenfunction expansion to only the first two eigenfunctions. The spectral density $S_{\delta n}(\Omega)$ of the operator $\delta\hat{n}(T)$ can be obtained by replacing $\Delta\hat{n}(T)$ with $\delta\hat{n}(T)$ in (65) and keeping only the $k=0$ term in the summation. This procedure preserves the commutator $[\hat{c}_0(T), \hat{c}_0^\dagger(T)]$, and the resulting $S_{\delta n}(\Omega)$ is

$$\begin{aligned}\frac{S_{\delta n}(\Omega)}{n_o} &= \frac{\tau_p^2}{4n_o}(R_{\rm nr}+R_{\rm pump})|H(\Omega)|^2 \\ &+ \frac{\tau_p}{2}\left[(2n_{\rm sp}-1)\left(\Omega^2+\frac{1}{\tau_{\rm nr}^2}\right)\tau_{\rm st}^2 \right. \\ &\left. + (\Omega^2+\gamma^2)\tau_{\rm st}^2\right]|H(\Omega)|^2\sqrt{1+\beta^2}. \quad (71)\end{aligned}$$

The first term is the contribution from the nonradiative carrier recombination noise and the noise in the laser pump. The second term is the noise contribution from the gain medium and vacuum field fluctuations. The term $\sqrt{1+\beta^2}$ is the excess noise factor and equals $1/|\int dt\, A_0{}^2(t)|^2$. The spectral density $S_{\Delta n}(\Omega)$ of the operator $\Delta\hat{n}(T)$ can be determined by solving (27), (63), and (65), and using the relation in (49) to yield

$$\begin{aligned}\frac{S_{\Delta n}(\Omega)}{n_o} &= \frac{\tau_p^2}{4n_o}(R_{\rm nr}+R_{\rm pump})|H(\Omega)|^2 \\ &+ \frac{(2n_{\rm sp}-1)}{\sqrt{1+\beta^2}}\sum_{k,q=0}^{N_{\rm max}-1} M_{k0}^{-1}M_{qk}M_{0q}^{-1}F_{qk}(\Omega) \\ &+ \frac{1}{\sqrt{1+\beta^2}}\sum_{k,q=0}^{N_{\rm max}-1} M_{k0}^{-1}M_{qk}M_{0q}^{-1}G_{qk}(\Omega). \quad (72)\end{aligned}$$

Expressions for the functions $F_{qk}(\Omega)$ and $G_{qk}(\Omega)$ are given in Appendix III. The two series in (72) can be summed exactly when $\Omega \approx 0$, and we obtain

$$\begin{aligned}&\frac{S_{\Delta n}(\Omega=0)}{n_o} \\ &= \frac{\tau_p^2}{4n_o}(R_{\rm nr}+R_{\rm pump}) \\ &+ \frac{\tau_p}{2}\left[(2n_{\rm sp}-1)\left(\frac{\tau_{\rm st}}{\tau_{\rm nr}}\right)^2+\left(\frac{\tau_{\rm st}}{\tau_{\rm nr}}\right)^2\right]\sqrt{1+\beta^2} \\ &+ \frac{\tau_p}{2}\left[2\left(\frac{\tau_{\rm st}}{\tau_{\rm nr}}\right)+1\right]. \quad (73)\end{aligned}$$

Analytical expression for $S_{\Delta n}(\Omega)$ can also be obtained when $\Omega \gg |4\lambda_0|$ as follows:

$$\begin{aligned}\frac{S_{\Delta n}(\Omega)}{n_o} &= \frac{\tau_p^2}{4n_o}(R_{\rm nr}+R_{\rm pump})|H(\Omega)|^2 \\ &+ \frac{\tau_p}{2}\left[(2n_{\rm sp}-1)\left(\Omega^2+\frac{1}{\tau_{\rm nr}^2}\right)\tau_{\rm st}^2 \right. \\ &\left. + (\Omega^2+\gamma^2)\tau_{\rm st}^2\right]|H(\Omega)|^2. \quad (74)\end{aligned}$$

The expression in (74) is similar to that in (71) but without the excess noise factor. Thus, like the pulse timing and frequency noise, the photon number noise also does not exhibit excess noise at frequencies much higher than $|4\lambda_0|$. Fig. 9 shows the spectral densities $S_{\Delta n}(\Omega)$ and $S_{\delta n}(\Omega)$ normalized to the expression in (71) without the excess noise factor for a semiconductor mode-locked laser. The values of the laser parameters are given in Table I. $R_{\rm nr}$ and $R_{\rm pump}$ are both assumed to be zero. In semiconductor mode-locked lasers, the values of the relaxation oscillation frequency $\Omega_{\rm ro}$ and the modulation damping constant γ are usually much larger than the value of $|2\lambda_0|$ (see Table I). The excess noise factor increases the low-frequency noise in $S_{\delta n}(\Omega)$ by $\sqrt{1+\beta^2}$ [see (71)]. The inclusion of the noise contributions from the higher order eigenfunctions in $S_{\Delta n}(\Omega)$ reduces the low-frequency photon number noise and partially compensates for the excess noise [see (73)]. At high frequencies ($\Omega \gg |4\lambda_0|$), the excess noise is absent. In actual semiconductor lasers, $R_{\rm nr}$ and $R_{\rm pump}$ cannot be assumed to be zero. $R_{\rm nr}$ and $R_{\rm pump}$ may contribute more to the pulse photon number noise than spontaneous emission and vacuum fluctuations, in which case the features in Fig. 9 due to the excess noise may be difficult to observe in experiments.

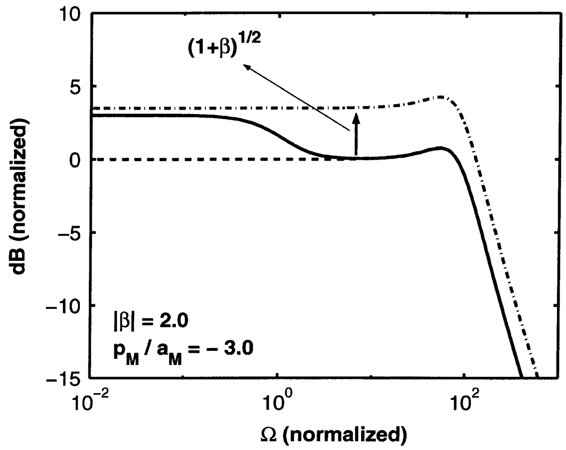

Fig. 9. Photon number noise spectral densities $S_{\delta n}(\Omega)$ (dash–dotted line) and $S_{\Delta n}(\Omega)$ are shown for $\beta = 2.0$ and $p_M/a_M = -3.0$ for an actively mode-locked semiconductor laser. The dashed line shows $S_{\delta n}(\Omega)/(1+\beta^2)^{1/2}$. The values of the laser parameters are given in Table I. $R_{\rm nr}$ and $R_{\rm pump}$ are both assumed to be zero. The spectral densities in the figure have been normalized w.r.t. the value $S_{\delta n}(\Omega=0)/(1+\beta^2)^{1/2}$. The frequency Ω has been normalized w.r.t. the value $|4\lambda_0|$. The resonance peak is due to the laser relaxation oscillations.

B. Phase Fluctuations

The equation for the operator $\delta\hat{\theta}(T)$ can be obtained from (63) as

$$\begin{aligned}\frac{d\delta\hat{\theta}(T)}{dT} &= -\frac{\alpha}{2\tau_{\rm st}n_o}\Delta\hat{N}(T) \\ &+ \frac{1}{2j}\left[\frac{\int dt\,(\hat{F}_g(t,T)+\hat{F}_v(t,T))A_0(t)}{\sqrt{n_o}\int dt\, A_0^2(t)} - \text{h.c.}\right]. \quad (75)\end{aligned}$$

The spectral density $S_{\delta\theta}(\Omega)$ of $\delta\hat{\theta}(T)$ can be found from (65) and (75) as

$$\begin{aligned}S_{\delta\theta}(\Omega) &= \alpha^2\frac{\tau_l^2}{16n_o}(R_{\rm nr}+R_{\rm pump})|H(\Omega)|^2 \\ &+ \left(\frac{n_{\rm sp}}{n_o}\right)\frac{1}{\tau_p}\left[\frac{1}{\Omega^2}+\alpha^2|H(\Omega)|^2\left(\frac{1}{\Omega^2}+\frac{\tau_p^2}{4}\right)\right] \\ &\times\sqrt{1+\beta^2}. \quad (76)\end{aligned}$$

TABLE I
LASER PARAMETERS USED IN FIG. 9 FROM [15]

Parameter	Value
Pulse photon number n_o	10^6
Pulse roundrip time T_R	0.2 ns
Pule repetition rate $1/T_R$	5.0 GHz
Photon loss τ_l	0.34 ns
Pulse width τ	5.0 ps
Pulse chirp magnitude $\|\beta\|$	2.0
Non-radiative recombination lifetime τ_{nr}	0.55 ns
Stimulated emission lifetime τ_{st}	1.95 ns
Laser relaxation oscillation frequency $\Omega_{ro}/2\pi$	275 MHz
Damping frequency $\|2\lambda_0\|/2\pi$	1.6 MHz
Spontaneous emission factor n_{sp}	1.5

The factor $\sqrt{1+\beta^2}$ in the above equation is the excess noise factor and equals $1/|\int dt\, A_0{}^2(t)|^2$. The spectral density $S_{\Delta\theta}(\Omega)$ of the pulse phase noise $\Delta\theta(T)$ can be obtained by solving (27), (63), and (65) and using the relation in (49) to yield

$$\begin{aligned} S_{\Delta\theta}(\Omega) &= \alpha^2 \frac{\tau_p^2}{16 n_o}(R_{\rm nr} + R_{\rm pump})|H(\Omega)|^2 \\ &\quad + \frac{(2n_{\rm sp}-1)}{n_o\sqrt{1+\beta^2}} \sum_{k,q=0}^{N_{\max}-1} M_{k0}^{-1} M_{qk} M_{0q}^{-1} U_{qk}(\Omega) \\ &\quad + \frac{1}{n_o\sqrt{1+\beta^2}} \sum_{k,q=0}^{N_{\max}-1} M_{k0}^{-1} M_{qk} M_{0q}^{-1} V_{qk}(\Omega). \end{aligned} \tag{77}$$

The expressions for the functions $U_{qk}(\Omega)$ and $V_{qk}(\Omega)$ are given in Appendix IV. Simple analytical expressions for $S_{\Delta\theta}(\Omega)$ can be obtained in two cases. For frequencies much lower than $|4\lambda_0|$ and the laser relaxation oscillation frequency $\Omega_{\rm ro}$, we obtain

$$\begin{aligned} S_{\Delta\theta}(\Omega \sim 0) &\approx S_{\delta\theta}(\Omega \sim 0) \\ &\approx \left(\frac{n_{\rm sp}}{n_o}\right) \frac{(1+\alpha^2)}{\tau_p \Omega^2} \sqrt{1+\beta^2}. \end{aligned} \tag{78}$$

Unlike pulse timing, frequency, and photon number fluctuations, pulse phase fluctuations at low frequencies exhibit the full excess noise even when the noise contribution from the higher order eigenfunctions is included. This is because the pulse phase noise $\Delta\hat{\theta}(T)$ at low frequencies is dominated by the noise contribution from the first eigenfunction, and this noise contribution is not damped and executes a random walk. From (78), it follows that the pulse phase diffusion at large time scales can be expressed in the time domain as

$$\begin{aligned} &\langle[\Delta\theta(T) - \Delta\theta(T')]^2\rangle \\ &\approx \langle[\delta\theta(T) - \delta\theta(T')]^2\rangle \\ &\approx \left(\frac{n_{\rm sp}}{n_o}\right) \frac{(1+\alpha^2)}{\tau_p} \sqrt{1+\beta^2}|T-T'|. \end{aligned} \tag{79}$$

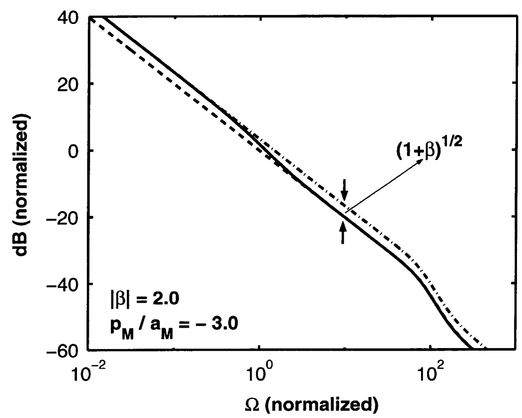

Fig. 10. The phase noise spectral densities $S_{\delta\theta}(\Omega)$ (dash–dotted line) and $S_{\Delta\theta}(\Omega)$ (solid line) are shown for $\beta = 2.0, p_M/a_M = -3.0$, and $\alpha = 3.0$ for an actively mode-locked semiconductor laser. The dashed line shows $S_{\delta\theta}(\Omega)/(1+\beta^2)^{1/2}$. The laser parameters are given in Table I. $R_{\rm nr}$ and $R_{\rm pump}$ are both assumed to be zero. The spectral densities in the figure have been normalized w.r.t. the value $S_{\delta\theta}(\Omega = |4\lambda_0|)/(1+\beta^2)^{1/2}$. The frequency Ω has been normalized w.r.t the value $|4\lambda_0|$.

When $\Omega \gg |4\lambda_0|$, a result similar to (76) is obtained but without the excess noise factor, shown as follows:

$$\begin{aligned} S_{\Delta\theta}(\Omega \gg |4\lambda_0|) &= \alpha^2 \frac{\tau_p^2}{16 n_o}(R_{\rm nr} + R_{\rm pump})|H(\Omega)|^2 \\ &\quad + \left(\frac{n_{\rm sp}}{n_o}\right)\frac{1}{\tau_p}\left[\frac{1}{\Omega^2} + \alpha^2 |H(\Omega)|^2 \left(\frac{1}{\Omega^2} + \frac{\tau_p^2}{4}\right)\right]. \end{aligned} \tag{80}$$

Thus, the excess noise is absent at high frequencies. Fig. 10 shows the phase noise spectral densities $S_{\Delta\theta}(\Omega)$ and $S_{\delta\theta}(\Omega)$, both normalized to the expression in (76) without the excess noise factor, for $\beta = 2.0, p_M/a_M = 3.0$, and $\alpha = 3.0$. $S_{\Delta\theta}(\Omega)$ shows the full excess noise factor $\sqrt{1+\beta^2}$ at low frequencies. At frequencies higher than $|4\lambda_0|$, the excess noise disappears.

VI. NOISE IN THE PULSES OUTSIDE THE LASER CAVITY

The master equation (2) describes the temporal evolution of a single pulse inside the laser cavity. The noise in the pulses coming out of the laser cavity is not the same as the noise in the pulse inside the laser cavity. The noise in the pulses outside the cavity is affected by the reflected vacuum fluctuations [23].

The rate of photon loss in the laser cavity can be expressed in the form

$$\frac{1}{\tau_l} = \frac{1}{\tau_{\rm li}} + \frac{1}{\tau_{\rm lo}} \tag{81}$$

where $\tau_{\rm lo}$ represents the rate of photon loss from the output coupler, and $\tau_{\rm li}$ describes the rate of photon loss due to other mechanisms inside the cavity. The noise operator for the vacuum fluctuations $\hat{F}_v(t,T)$ in the master equation (2) can be written as

$$\hat{F}_v(t,T) = \hat{F}_{\rm vi}(t,T) + \hat{F}_{\rm vo}(t,T). \tag{82}$$

Assuming that $n_{\rm th} = 0$, the only nonzero correlations of the noise operators $\hat{F}_{\rm vi}(t,T)$ and $\hat{F}_{\rm vo}(t,T)$ are

$$\langle \hat{F}_{\rm vi}(t,T)\hat{F}^\dagger_{\rm vi}(t',T')\rangle = \left[\frac{2}{\tau_{\rm li}} - O(t) - O^\dagger(t)\right]\delta(t-t')\delta(T-T') \tag{83}$$

$$\langle \hat{F}_{\rm vo}(t,T)\hat{F}^\dagger_{\rm vo}(t',T')\rangle = \frac{2}{\tau_{\rm lo}}\delta(t-t')\delta(T-T'). \tag{84}$$

The delta function $\delta(T-T')$ in (84) implies that the noise added to the pulse in different round trips is uncorrelated. Therefore, the correlation function in (84) for equal times must be interpreted as

$$\langle \hat{F}_{\rm vo}(t,T)\hat{F}^\dagger_{\rm vo}(t',T)\rangle = \frac{2}{T_R\tau_{\rm lo}}\delta(t-t'). \tag{85}$$

The pulses coming out of the laser cavity are more appropriately described by labeling them with a discrete index. A brief review of the discrete time Fourier transforms and the associated noise spectral densities is given in Appendix V.B. The field operator of the mth pulse which comes out of the laser at time $T = mT_R$ is assumed to be $\hat{\phi}(t,m)$ and is related to $\hat{\phi}(t,T)$ as

$$\hat{\phi}(t,m) = \sqrt{\frac{2T_R}{\tau_{\rm lo}}}\hat{\phi}(t,T=mT_R) - \sqrt{\frac{T_R\tau_{\rm lo}}{2}}\hat{F}_{\rm vo}(t,T=mT_R). \tag{86}$$

The second term on the right hand side represents the reflected vacuum fluctuations. $\hat{\phi}(t,m)$ obeys the commutation relation

$$[\hat{\phi}(t,m), \hat{\phi}^\dagger(t',m)] = \delta(t-t'). \tag{87}$$

$\hat{\phi}(t,m)$ can also be expanded in terms of the functions $A_k(t)$ as follows:

$$\hat{\phi}(t,m) = \sqrt{n_o'}A_0(t) + \sqrt{n_o'}\sum_{k=0}^{\infty}\hat{d}_k(m)A_k(t) \tag{88}$$

where the average number n_o' of photons in the output pulses equals $(2T_R/\tau_{\rm lo})n_o$. Equations (87) and (88) give the commutation relations for the operators $\hat{d}_k(m)$ as follows:

$$[\hat{d}_k(m), \hat{d}^\dagger_q(m)] = \frac{1}{n_o'}\frac{\int dt\, A^*_q(t)A_k(t)}{\int dt\, A^{*2}_q(t)\int dt\, A^2_k(t)}. \tag{89}$$

Equations (86) and (88) give

$$\hat{d}_k(m) = \hat{c}_k(T=mT_R) - \frac{\tau_{\rm lo}}{2\sqrt{n_o}}\frac{\int dt\, \hat{F}_{\rm Vo}(t,T=mT_R)A_k(t)}{\int dt\, A^2_k(t)}. \tag{90}$$

The expression above satisfies the commutation relation for $\hat{d}_k(m)$ given in (89). The operators for the noise in the output pulses can be expressed in terms of $\hat{d}_k(m)$ if the substitutions $\hat{c}_k(T) \to \hat{d}_k(m)$ and $n_o \to n_o'$ are made in (41)–(44). It should be noted that the operators $\Delta\hat{n}(m), \Delta\hat{\theta}(m), \Delta\hat{t}(m)$, and $\Delta\hat{\omega}(m)$ for the noise in the output pulses are functions of the discrete index m, and the corresponding noise spectral densities $\Phi_{\Delta n}(\Omega T_R), \Phi_{\Delta\theta}(\Omega T_R), \Phi_{\Delta J}(\Omega T_R)$, and $\Phi_{\Delta\omega}(\Omega T_R)$ are periodic functions of Ω with a period $2\pi/T_R$ (see Appendix V-B).

A. Timing and Frequency Fluctuations in the Output Pulses

The spectral density $\Phi_{\Delta J}(\Omega T_R)$ of the timing noise $\Delta\hat{t}(m)$ in the output pulses can be determined using the methods described in Section IV, and we obtain

$$\Phi_{\Delta J}(\Omega T_R) = \frac{1}{T_R}\sum_{m=-\infty}^{\infty} S_{\Delta J}\left(\Omega - \frac{2\pi m}{T_R}\right) + \frac{\tau^2}{2n_o}\frac{1}{T_R}\sum_{m=-\infty}^{\infty} R\left(\Omega - \frac{2\pi m}{T_R}\right) + \frac{\tau^2}{2n_o'} \tag{91}$$

where the function $R(\Omega)$ is

$$R(\Omega) = \frac{(2\lambda_0 + 2\lambda_0^*)(\Omega^2 + |2\lambda_0|^2)}{(\Omega^2 - |2\lambda_0|^2)^2 + \Omega^2(-2\lambda_0 - 2\lambda_0^*)^2} \tag{92}$$

and $S_{\Delta J}(\Omega)$ is given by the expression in (60). The last two terms on the right-hand side in (91) are due to the reflected vacuum fluctuations. The mean square value $\langle \Delta\hat{t}^2(m)\rangle$ of the timing noise is

$$\langle \Delta\hat{t}^2(m)\rangle = \langle \Delta\hat{t}^2(T)\rangle + \frac{\tau^2}{2n_o'}\left(1 - \frac{n_o'}{n_o}\right). \tag{93}$$

From the value of $\langle \Delta\hat{t}^2(T)\rangle$ given in (51), it follows that the reflected vacuum fluctuations make a negligible contribution to the mean square timing noise in the output pulses when $-{\rm Re}(\lambda_0)\tau_p \ll 1$. The spectral density $\Phi_{\Delta\omega}(\Omega T_R)$ and the mean square value $\langle \Delta\hat{\omega}^2(m)\rangle$ of the pulse frequency fluctuations are related to those of the pulse timing fluctuations by a constant factor $(1+\beta^2)/\tau^4$.

B. Photon Number and Phase Fluctuations in the Output Pulses

The spectral density $\Phi_{\Delta n}(\Omega T_R)$ of the photon number noise $\Delta\hat{n}(m)$ in the output pulses can be determined using the methods described in Section V, and we obtain

$$\frac{\Phi_{\Delta n}(\Omega T_R)}{n_o'} = \frac{1}{T_R}\left(\frac{n_o'}{n_o}\right)\sum_{m=-\infty}^{\infty}\frac{S_{\Delta n}\left(\Omega - \frac{2\pi m}{T_R}\right)}{n_o} + \left[1 - \frac{\gamma\tau_{\rm st}\tau_p}{T_R}\left(\frac{n_o'}{n_o}\right)\sum_{m=-\infty}^{\infty}\left|H\left(\Omega - \frac{2\pi m}{T_R}\right)\right|^2\right] \tag{94}$$

where $S_{\Delta n}(\Omega)$ is given in (72). It follows from the expression above that the mean square value $\langle \Delta \hat{n}^2(m)\rangle$ of the photon number fluctuations in the output pulses is

$$\frac{\langle \Delta \hat{n}^2(m)\rangle}{n'_o} = \left(\frac{n'_o}{n_o}\right) \frac{\langle \Delta \hat{n}^2(T)\rangle}{n_o} + \left(1 - \frac{n'_o}{n_o}\right). \tag{95}$$

The second term on the right-hand side is due to the reflected vacuum fluctuations.

The spectral density $\Phi_{\Delta\theta}(\Omega T_R)$ of the pulse phase noise $\Delta\hat{\theta}(m)$ is

$$\begin{aligned}\Phi_{\Delta\theta}(\Omega T_R) &= \frac{1}{T_R} \sum_{m=-\infty}^{\infty} S_{\Delta\theta}\left(\Omega - \frac{2\pi m}{T_R}\right) \\ &+ \frac{1}{4n'_o}\left[1 - \frac{1}{T_R}\left(\frac{n'_o}{n_o}\right) \sum_{m=-\infty}^{\infty} 2\pi\delta\left(\Omega - \frac{2\pi m}{T_R}\right)\right]\end{aligned} \tag{96}$$

where $S_{\Delta\theta}(\Omega)$ is given by the expression in (77). The last term on the right-hand side in (96) is due to the reflected vacuum fluctuations.

VII. Pulse Fluctuation Operators and Noise Measurements

A widely used technique for characterizing the photon number and timing fluctuations of pulses from mode-locked lasers is measuring the spectral density of the photodetector current noise [17]–[19], [25]. Here, it is shown that the photodetector current noise spectral density is directly related to the spectral densities $\Phi_{\Delta n}(\Omega T_R)$ and $\Phi_{\Delta J}(\Omega T_R)$ of the pulse photon number and timing noise operators $\Delta\hat{n}(m)$ and $\Delta\hat{t}(m)$, respectively, as defined in this paper. The operator for the photodetector current is

$$\hat{I}(t) = e \sum_{m=-\infty}^{\infty} \hat{\phi}^{\dagger}(t - mT_R, m)\hat{\phi}(t - mT_R, m) \tag{97}$$

where e is the electron charge and $\hat{\phi}(t, m)$ is the field operator for the mth output pulse from the laser. The photodetector is assumed to have a unit quantum efficiency and a response time much faster than the frequencies of interest. If necessary, the frequency response of the detector can be taken into account by multiplying the spectral density of the current by the detector frequency response function [26]. Equation (97) can be written as

$$\begin{aligned}\hat{I}(t) &= en'_o \sum_{m=-\infty}^{\infty} A_0^*(t - mT_R)A_0(t - mT_R) \\ &+ e\sqrt{n'_o} \sum_{m=-\infty}^{\infty} [A_0^*(t - mT_R)\hat{\psi}(t - mT_R, m) + \text{h.c.}].\end{aligned} \tag{98}$$

The second term in the above equation is the current noise $\Delta\hat{I}(t)$. The spectral density $S_{\Delta I}(\Omega)$ of the current noise is defined in terms of the symmetric time averaged correlation function (see Appendix V-A) as

$$\begin{aligned}S_{\Delta I}(\Omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} dt'\, \frac{1}{2}[\langle \Delta\hat{I}(t)\Delta\hat{I}(t + t')\rangle \\ + \langle \Delta\hat{I}(t + t')\Delta\hat{I}(t)\rangle] \exp(-j\Omega t').\end{aligned} \tag{99}$$

After some algebra, (99) yields

$$\begin{aligned}S_{\Delta I}(\Omega) \approx \frac{e^2}{T_R}\{\Phi_{\Delta n}(\Omega T_R) + \Omega^2 n'^2_o \Phi_{\Delta J}(\Omega T_R) \\ + j\Omega n'_o[\Phi_{\Delta J\Delta n}(\Omega T_R) - \Phi_{\Delta J\Delta n}(-\Omega T_R)]\}.\end{aligned} \tag{100}$$

The spectral density $\Phi_{\Delta J\Delta n}(\Omega T_R)$ is the discrete time Fourier transform of the symmetric cross-correlation function between $\Delta\hat{t}(m)$ and $\Delta\hat{n}(m)$. At small frequencies, $S_{\Delta I}(\Omega)$ is proportional to the spectral density $\Phi_{\Delta n}(\Omega T_R)$ of the pulse photon number fluctuations. Assuming no correlation between the pulse photon number noise and timing noise, the pulse timing noise spectral density $\Phi_{\Delta J}(\Omega T_R)$ can be obtained by measuring the current noise spectral density near a large harmonic of the pulse repetition frequency where the photon number noise is negligible. The timing noise spectral density $\Phi_{\Delta J}(\Omega T_R)$ can be obtained more reliably by mixing the photodetector current with a signal from the same RF oscillator that provides the active modulation for the mode-locked laser. In this case, the timing noise is measured relative to the timing (or phase) noise of the RF oscillator. By appropriately adjusting the phase of the signal before mixing it with the photodetector current, the contribution from the pulse photon number fluctuations can be removed [17]–[19], [26].

VIII. Conclusion

A quantum theory for the noise of optical pulses in actively mode-locked lasers with phase modulation and/or group velocity dispersion was presented. Quantum operators were constructed for the pulse photon number, phase, timing, and frequency noise. It was shown that when the linear operator that describes the time evolution of the pulse fluctuations inside the laser cavity is not normal (or Hermitian) the pulse photon number, phase, timing, and frequency fluctuations exhibit excess noise. The excess noise was found to appear only at low frequencies in the spectral densities of the pulse photon number, phase, timing, and frequency noise operators. Finally, a connection was made with experiments that measure pulse noise by measuring the noise in the photodetector current. It was shown that the fluctuations described by the quantum operators constructed in Section III-C are measured in these experiments.

The model for the actively mode-locked laser discussed in this paper is linear. Future work in this direction needs to address the effect of nonlinearities, such as dynamic gain or loss saturation and dynamic self-phase modulation, on the pulse noise. These dynamic nonlinearities can make the operator that describes the time evolution of the pulse fluctuations non-Hermitian. Finally, it is not clear if the minimum-error expansion used to establish the result in (49) is optimum in the sense that it gives the fastest rate of convergence. The authors believe that a suitable orthonormal expansion may provide faster convergence, but this remains to be explored.

APPENDIX I
PROPERTIES OF COMPLEX HERMITE–GAUSSIANS

The complex Hermite–Gaussians $A_k(t)$ are defined as

$$A_k(t) = B_k H_k\left(\frac{t}{\tau}\sqrt{1+j\beta}\right)\exp\left[-\frac{t^2}{2\tau^2}(1+j\beta)\right] \quad (101)$$

where B_k is a normalization constant and $H_k(\cdot)$ is the kth Hermite polynomial [24]. If $A_k(t)$ is normalized such that $\int dt\,|A_k(t)|^2 = 1$, then B_k is

$$\frac{1}{B_k^2} = 2^k k!\sqrt{\pi}\tau P_k(\sqrt{1+\beta^2}) \quad (102)$$

where $P_k(\cdot)$ is the kth Legendre polynomial [24]. The following integrals have been used in this paper:

$$\int dt\,A_k^2(t) = \frac{(1-j\beta)^{1/2}}{\sqrt{1+\beta^2}P_k(\sqrt{1+\beta^2})} \quad (103)$$

$$M_{qk} = M_{kq}^* = \int dt\,A_q^*(t)A_k(t)$$
$$= \sqrt{\frac{k!}{q!}}\frac{P_{\frac{q+k}{2}}^{\frac{q-k}{2}}(\sqrt{1+\beta^2})}{\sqrt{P_q(\sqrt{1+\beta^2})P_k(\sqrt{1+\beta^2})}} \quad (104)$$

where $P_{(q+k)/(2)}^{(q-k)/(2)}(\cdot)$ is the associated Legendre function with the following properties:

$$P_{\frac{q+k}{2}}^{\frac{q-k}{2}}(\cdot) = (-1)^{\frac{k-q}{2}}\frac{q!}{k!}P_{\frac{k+q}{2}}^{\frac{k-q}{2}}(\cdot) \quad (105)$$

$$P_k^0(\cdot) = P_k(\cdot). \quad (106)$$

APPENDIX II
DIVERGENCE OF THE CONVENTIONAL PERTURBATIVE EXPANSION AND THE MINIMUM ERROR EXPANSION

In Section IV, it was mentioned that a series of the form

$$\sum_{k,q=0}^{\infty} F_{qk}\frac{M_{pk}M_{qk}M_{qp}}{\int dt\,A_q^{*2}(t)\int dt\,A_k^2(t)} \quad (107)$$

(where $F_{qk}\ (= F_{kq}^*)$ decays only algebraically as k, q become large) does not converge when only a finite number of terms are included in the summation. The divergence is best illustrated by assuming $F_{qk} = 1$. In this case, the series can be summed exactly using the completeness relation for the eigenfunctions in (21) as

$$\sum_{k,q=0}^{\infty}\frac{M_{pk}M_{qk}M_{qp}}{\int dt\,A_q^{*2}(t)\int dt\,A_k^2(t)} = 1. \quad (108)$$

Fig. 11 shows the result when $p = 1$ and only terms in which $k, q < N_{\max}$ are included in the summation in (108)). When

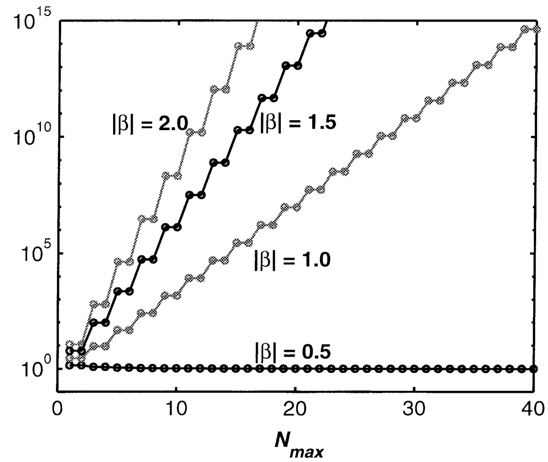

Fig. 11. The result obtained when $p = 1$ and only terms in which $k, q < N_{\max}$ are included in the summation in (108). The series diverges when $|\beta|$ is larger than $\beta_c = 1/\sqrt{3}$. The steps appear because when p is odd only those terms in which both k, q are odd contribute to the result.

$|\beta|$ is small, the series converges to unity. When $|\beta|$ is large, the series diverges exponentially as $N_{\max}$ increases. The largest terms in the series are the diagonal $(k = q)$ terms. From the properties of the eigenfunctions $A_k(t)$ in Appendix I the ratio of two successive diagonal terms can be calculated as

$$\lim_{k\to\infty}\frac{|\int dt\,A_p^*(t)A_{k+2}(t)|^2}{\left|\int dt\,A_{k+2}^2(t)\right|^2}\frac{\left|\int dt\,A_k^2(t)\right|^2}{|\int dt\,A_p^*(t)A_k(t)|^2} \approx \beta^2(|\beta|+\sqrt{1+\beta^2})^2. \quad (109)$$

The series diverges when $|\beta| > \beta_c$, where the critical value β_c is determined by setting $\beta(|\beta|+\sqrt{1+\beta^2})$ equal to unity. This yields $\beta_c = 1/\sqrt{3} \approx 0.577$. This critical value was found in [21] in the general context of series expansions using complex Hermite–Gaussians. Numerical calculations confirm the value of $1/\sqrt{3}$ for β_c. Since the exact sum is not infinite, it follows that, when $N_{\max}$ is infinitely large, the off-diagonal $(k \neq q)$ terms in the series suppress the divergence coming from the diagonal $(k = q)$ terms.

The physical significance of this result is that, when the magnitude of the pulse chirp is larger than β_c, although the noise in the eigenfunctions is large, the noise in different eigenfunctions is highly correlated. These noise correlations suppress the divergence in the perturbative expansion for the pulse noise provided an infinite number of eigenfunctions are included in the perturbative expansion. The divergence is not physical and appears only because the eigenfunction basis is not a suitable basis for studying the pulse noise when $|\beta| > \beta_c$. Below, a technique to obtain a convergent expansion for series of the form (107) is presented.

A. Minimum-Error Series Expansion

Since the complex Hermite–Gaussians $A_k(t)$ form a complete set, an arbitrary complex function $f(t)$ can be expanded in a series of the form

$$f(t) = \sum_{k=0}^{\infty} c_k A_k(t). \quad (110)$$

The values of the coefficients c_k can be obtained by projecting the function $f(t)$ onto $A_k(t)$ using the functions $A_k^*(t)$ as

$$c_k = \frac{\int dt\, A_k(t) f(t)}{\int dt\, A_k^2(t)}. \tag{111}$$

However, there is no guarantee that a series obtained this way converges in finite steps for all functions $f(t)$. In [21], it was pointed out that when $f(t)$ equals $A_0^*(t)$ the series fails to converge when the magnitude of the chirp parameter β is larger than β_c $(= 1/\sqrt{3})$. In [21], it was also shown that the series obtained by choosing the values of c_k such that the mean square error

$$\int dt \left| f(t) - \sum_{k=0}^{N_{\max}-1} c_k A_k(t) \right|^2 \tag{112}$$

is minimized converges even when $|\beta| > \beta_c$. This minimum-error expansion is used here to obtain a convergent expansion for series of the type (107). We define Q_p as

$$Q_p = \int dt\, A_p^*(t) f(t) \tag{113}$$

$$= \sum_{k=0}^{\infty} M_{pk} c_k. \tag{114}$$

The expansion for $f(t)$ given by (110) has been used in (114). If it is assumed that $f(t)$ is such that

$$c_q^* c_k = F_{qk} \frac{M_{qk}}{\int dt\, A_q^{*2}(t) \int dt\, A_k^2(t)} \tag{115}$$

then a series of the form (107) is generated when $|Q_p|^2$ is evaluated. The function $f(t)$ can also be expanded using the complete set formed by the eigenfunctions $A_k^*(t)$ of the adjoint operator

$$f(t) = \sum_{k=0}^{\infty} b_k A_k^*(t). \tag{116}$$

Using the expansion above, the following expression for Q_p is simple:

$$Q_p = b_p \int dt\, A_p^{*2}(t). \tag{117}$$

We need to find optimal values of the coefficients b_k when the series expansion in (116) is restricted to only $N_{\max}$ terms. If the value of $N_{\max}$ is finite, the resulting series can only be approximate, and this approximation will be considered good if the series converges as $N_{\max}$ becomes large. For a given $N_{\max}$, we choose b_k to minimize the mean square error

$$\int dt \left| f(t) - \sum_{k=0}^{N_{\max}-1} b_k A_k^*(t) \right|^2 \tag{118}$$

which gives

$$\sum_{k=0}^{N_{\max}-1} b_k \int dt\, A_k^*(t) A_q(t) = \int dt\, A_q(t) f(t) \tag{119}$$

$$\sum_{k=0}^{N_{\max}-1} M_{kq} b_k = c_q \int dt\, A_q^2(t). \tag{120}$$

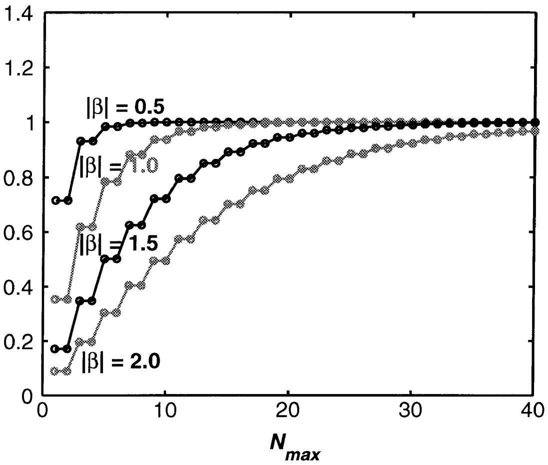

Fig. 12. Result obtained when $p = 1$ and $F_{qk} = 1$ in the series on the right-hand side in (126) for different values of $N_{\max}$. The series converges for all values of β. The steps appear because when p is odd only those terms in which both k, q are odd contribute to the result.

Inverting the matrix relation in the above equation gives the following desired result:

$$b_k = \sum_{q=0}^{N_{\max}-1} M_{qk}^{-1} c_q \int dt\, A_q^2(t). \tag{121}$$

Using (114), (115), (117), and (121), we obtain

$$|Q_p|^2 = \sum_{k,q=0}^{\infty} M_{pk} M_{qp} (c_q^* c_k) \tag{122}$$

$$= \sum_{k,q=0}^{\infty} F_{qk} \frac{M_{pk} M_{qk} M_{qp}}{\int dt\, A_q^{*2}(t) \int dt\, A_k^2(t)} \tag{123}$$

$$= \left| \int dt\, A_p^2(t) \right|^2 (b_p^* b_p) \tag{124}$$

$$= \lim_{N_{\max}\to\infty} \left| \int dt\, A_p^2(t) \right|^2 \times \sum_{k,q=0}^{N_{\max}-1} F_{qk} M_{kp}^{-1} M_{qk} M_{pq}^{-1}. \tag{125}$$

Thus, one obtains the following relation:

$$\sum_{k,q=0}^{\infty} F_{qk} \frac{M_{pk} M_{qk} M_{qp}}{\int dt\, A_q^{*2}(t) \int dt\, A_k^2(t)} = \lim_{N_{\max}\to\infty} \left| \int dt\, A_p^2(t) \right|^2 \times \sum_{k,q=0}^{N_{\max}-1} F_{qk} M_{kp}^{-1} M_{qk} M_{pq}^{-1}. \tag{126}$$

The series on the right-hand side of the above equation converges when $|\beta| > \beta_c$. Fig. 12 shows the result when $p = 1$ and $F_{qk} = 1$. As $N_{\max}$ increases, the series converges to the exact value of unity [see (108)].

APPENDIX III
DEFINITIONS OF $F_{qk}(\Omega)$ AND $G_{qk}(\Omega)$

The functions $F_{qk}(\Omega)$ and $G_{qk}(\Omega)$ can be expressed in terms of $f_k(\Omega)$ and $g_k(\Omega)$ where

$$f_0(\Omega) = \tau_{\text{st}} \left(j\Omega + \frac{1}{\tau_{\text{nr}}} \right) H(\Omega) \quad (127)$$

$$g_0(\Omega) = \tau_{\text{st}}(j\Omega + \gamma)H(\Omega) \quad (128)$$

$$f_k(\Omega) = -\left(\frac{j\Omega + 2/\tau_p}{j\Omega - 2k\lambda_0} \right) H(\Omega) + \left(\frac{2/\tau_p}{j\Omega - 2k\lambda_0} \right), \quad k \geq 1 \quad (129)$$

$$g_k(\Omega) = -\left(\frac{2k\lambda_0 + 2/\tau_p}{j\Omega - 2k\lambda_0} \right) H(\Omega) + \left(\frac{2/\tau_p}{j\Omega - 2k\lambda_0} \right), \quad k \geq 1 \quad (130)$$

and,

$$F_{qk}(\Omega) = \frac{\tau_p}{4}[f_k(\Omega)f_q^*(\Omega) + (\Omega \to -\Omega)] \quad (131)$$

$$\begin{aligned} G_{qk}(\Omega) &= \frac{\tau_p}{4}[1 - (k\lambda_0 + q\lambda_0^*)\tau_p] \\ &\quad \times [g_k(\Omega)g_q^*(\Omega) + (\Omega \to -\Omega)] \\ &\approx \frac{\tau_p}{4}[g_k(\Omega)g_q^*(\Omega) + (\Omega \to -\Omega)]. \end{aligned} \quad (132)$$

APPENDIX IV
DEFINITIONS OF $U_{qk}(\Omega)$ AND $V_{qk}(\Omega)$

The functions $U_{qk}(\Omega)$ and $V_{qk}(\Omega)$ can be expressed in terms of the functions $u_k(\Omega)$ and $v_k(\Omega)$ where

$$u_k(\Omega) = \frac{\alpha}{2} \left(\frac{j\Omega + 2/\tau_p}{j\Omega - 2k\lambda_0} \right) H(\Omega) + \frac{1}{2j} \left(\frac{2/\tau_p}{j\Omega - 2k\lambda_0} \right) \quad (133)$$

$$v_k(\Omega) = \frac{\alpha}{2} \left(\frac{2k\lambda_0 + 2/\tau_p}{j\Omega - 2k\lambda_0} \right) H(\Omega) + \frac{1}{2j} \left(\frac{2/\tau_p}{j\Omega - 2k\lambda_0} \right) \quad (134)$$

and

$$U_{qk}(\Omega) = \frac{\tau_p}{4}[u_k(\Omega)u_q^*(\Omega) + (\Omega \to -\Omega)] \quad (135)$$

$$\begin{aligned} V_{qk}(\Omega) &= \frac{\tau_p}{4}[1 - (k\lambda_0 + q\lambda_0^*)\tau_p] \\ &\quad \times [v_k(\Omega)v_q^*(\Omega) + (\Omega \to -\Omega)] \\ &\approx \frac{\tau_p}{4}[v_k(\Omega)v_q^*(\Omega) + (\Omega \to -\Omega)]. \end{aligned} \quad (136)$$

APPENDIX V
FOURIER TRANSFORMS AND NOISE SPECTRAL DENSITIES

The properties of the continuous time and discrete time Fourier transforms and the corresponding noise spectral densities are briefly reviewed here.

A. Continuous Time Fourier Transform and Noise Spectral Densities

The Fourier transform of a zero mean noise operator $\hat{W}(T)$ is defined as

$$\hat{W}(\Omega) = \int dt\, \hat{W}(T) \exp(-j\Omega T). \quad (137)$$

The inverse Fourier transform is

$$\hat{W}(T) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \hat{W}(\Omega) \exp(j\Omega T). \quad (138)$$

The spectral density $S_W(\Omega)$ of $\hat{W}(T)$ is defined as the Fourier transforms of the symmetric correlation function

$$\begin{aligned} S_W(\Omega) = \int_{-\infty}^{\infty} ds\, \frac{1}{2}[&\langle \hat{W}(T)\hat{W}(T+s) \rangle \\ &+ \langle \hat{W}(T+s)\hat{W}(T) \rangle] \exp(-j\Omega s). \end{aligned} \quad (139)$$

It is assumed in the definition above that the correlation function is stationary and, therefore, independent of the time variable T. It follows from the definition of $S_W(\Omega)$ that the mean square value $\langle \hat{W}^2(T) \rangle$ of the fluctuations is

$$\langle \hat{W}^2(T) \rangle = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi}\, S_W(\Omega). \quad (140)$$

B. Discrete-Time Fourier Transforms and Noise Spectral Densities

The discrete-time Fourier transform $\hat{W}(\Omega T_R)$ of a zero mean noise operator $\hat{W}(m)$, which is a function of the discrete index m, is defined as

$$\hat{W}(\Omega T_R) = \sum_{m=-\infty}^{\infty} \hat{W}(m) \exp(-j\Omega T_R m). \quad (141)$$

$\hat{W}(\Omega T_R)$ is periodic in Ω with a period $2\pi/T_R$. The inverse Fourier transform is

$$\hat{W}(m) = T_R \int_{-\pi/T_R}^{\pi/T_R} \frac{d\Omega}{2\pi}\, \hat{W}(\Omega T_R) \exp(j\Omega T_R m). \quad (142)$$

The spectral density $\Phi_W(\Omega T_R)$ of $\hat{W}(m)$ is defined as the discrete-time Fourier transform of the symmetric correlation function

$$\begin{aligned} \Phi_W(\Omega T_R) = \sum_{m=-\infty}^{\infty} \frac{1}{2}[&\langle \hat{W}(n)\hat{W}(n+m) \rangle \\ &+ \langle \hat{W}(n+m)\hat{W}(n) \rangle] \exp(-j\Omega T_R m). \end{aligned} \quad (143)$$

It is assumed in the definition above that the correlation function is stationary and, therefore, independent of n. The spectral density $\Phi_W(\Omega T_R)$ is periodic in Ω with a period $2\pi/T_R$. It follows from the definition of $\Phi_W(\Omega T_R)$ that the mean square value $\langle \hat{W}^2(m) \rangle$ of the fluctuations is

$$\langle \hat{W}^2(m) \rangle = T_R \int_{-\pi/T_R}^{\pi/T_R} \frac{d\Omega}{2\pi} \Phi_W(\Omega T_R). \quad (144)$$

ACKNOWLEDGMENT

The authors acknowledge helpful discussions with E. P. Ippen, M. E. Grein, L. A. Jiang, P. W. Juodawlkis, and J. J. Hargreaves.

REFERENCES

[1] H. A. Haus, "A theory of forced mode locking," *IEEE J. Quantum Electron.*, vol. QE-11, pp. 323–330, 1975.

[2] H. A. Haus and A. Mecozzi, "Noise of mode-locked lasers," *IEEE J. Quantum Electron.*, vol. 29, pp. 983–996, 1993.

[3] H. A. Haus, M. Margalit, and C. X. Yu, "Quantum noise of a mode-locked laser," *J. Opt. Soc. Amer. B*, vol. 17, pp. 1240–1256, 2000.

[4] E. Zeidler, *Applied Functional Analysis*. New York: Springer-Verlag, 1995.

[5] K. Petermann, "Calculated spontaneous emission factor for double-heterstructure injection laser," *IEEE J. Quantum Electron.*, vol. QE-15, pp. 566–570, 1979.

[6] H. A. Haus and S. Kawakami, "On the 'excess spontaneous emission factor' in gain-guided laser amplifiers," *IEEE J. Quantum Electron.*, vol. QE-21, pp. 63–69, 1985.

[7] A. E. Siegman, "Excess spontaneous emission in non-Hermitian optical systems. I. Laser amplifiers," *Phys. Rev. A*, vol. 39, pp. 1253–1263, 1989.

[8] ——, "Excess spontaneous emission in non-Hermitian optical systems. II. Laser oscillators," *Phys. Rev. A*, vol. 39, pp. 1264–1268, 1989.

[9] S. Longhi and P. Laporta, "Excess noise in intracavity laser frequency modulation," *Phys. Rev. E*, vol. 61, pp. R989–R992, 2000.

[10] F. X. Kaertner, D. M. Zurnbuhl, and N. Matuscheck, "Turbulence in mode-locked lasers," *Phys. Rev. Letts.*, vol. 82, pp. 4428–4431, 1999.

[11] P. T. Ho, "Phase and amplitude fluctuations in a mode-locked laser," *IEEE J. Quantum Electron.*, vol. QE-21, pp. 1806–1813, 1985.

[12] D. R. Hjelme and A. R. Mickelson, "Theory of timing jitter in actively mode-locked lasers," *IEEE J. Quantum Electron.*, vol. 28, pp. 1594–1606, 1992.

[13] L. A. Jiang, M. E. Grein, H. A. Haus, and E. P. Ippen, "Noise of mode-locked semiconductor lasers," *IEEE J. Select. Topics Quantum Electron.*, vol. 7, pp. 159–167, 2001.

[14] D. J. Derickson, R. J. Helkey, A. Mar, J. R. Karin, J. G. Wasserbauer, and J. E. Bowers, "Short pulse generation using multisegment mode-locked semiconductor lasers," *IEEE J. Quantum Electron.*, vol. 28, pp. 2186–2202, 1992.

[15] F. Rana and R. J. Ram, "Noise and timing itter in active and hybrid mode-locked semiconductor lasers," in *Proc. Conf. Lasers and Electro-Optics*, Washington, DC, 2001, pp. 6–7.

[16] H. A. Haus and Y. Lai, "Quantum theory of soliton squeezing: A linearized approach," *J. Opt. Soc. Amer. B*, vol. 7, p. 386, 1990.

[17] D. J. Derickson, P. A. Morton, and J. E. Bowers, "Comparison of timing jitter in external and monolithic cavity mode-locked semiconductor lasers," *Appl. Phys. Lett.*, vol. 59, pp. 3372–3374, 1991.

[18] A. Braun, J. H. Abeles, C. M. DePriest, and P. J. Delfyett Jr., "10-GHz ultralow-noise optical sampling stream from a semiconductor diode ring laser," *IEEE Photon. Technol. Lett.*, vol. 13, pp. 1109–1111, 2001.

[19] J. J. Hargreaves, P. W. Juodawlkis, J. J. Plant, J. P. Donnelly, and J. C. Twichell, "Residual phase-noise measurements of actively mode-locked fiber and semiconductor lasers," in *Proc. Annu. Meeting IEEE Lasers and Electro-Optics Society*, 2001, pp. 115–116.

[20] L. A. Coldren and S. W. Corzine, *Diode Lasers and Photonic Integrated Circuits*. New York: Wiley, 1995.

[21] A. Kostenbauder and A. E. Siegman, "Eigenmode expansions using biorthogonal functions: Complex-valued Hermite–Gaussians," *J. Opt. Soc. Amer.*, vol. B14, pp. 1780–1790, 1997.

[22] D. F. Walls and G. J. Milburn, *Quantum Optics*. Berlin, Germany: Springer-Verlag, 1994.

[23] Y. Yamamoto, S. Machida, and O. Nilsson, "Amplitude squeezing in a pump-noise-suppressed laser oscillator," *Phys. Rev. A*, vol. 34, pp. 4025–4042, 1986.

[24] *Handbook of Applied Mathematics*, Van Nostrand-Reinhold, New York, 1974.

[25] D. Von Der Linde, "Characterization of the noise in continuously operating mode-locked lasers," *J. Appl. Phys. B*, vol. 39, pp. 201–217, 1986.

[26] F. Rana, H. L. T. Lee, R. J. Ram, M. E. Grein, L. A. Jiang, E. P. Ippen, and H. A. Haus, "Characterization of the noise and correlations in harmonically mode-locked lasers," *J. Opt. Soc. Amer. B*, vol. 19, pp. 2609–2621, 2002.

Farhan Rana was born in 1971 in Karachi, Pakistan. He received the B.S., M.S., and Ph.D. degrees, all in electrical enginnering, from the Massachusetts Institute of Technology (MIT), Cambridge.

He worked on a variety of different topics related to semiconductor optoelectronics, quantum optics, and mesoscopic physics during his Ph.D. research. Before starting his doctoral studies, he worked at IBM's T. J. Watson Research Center, Yorktown Heights, NY, on silicon nanocrystal and quantum dot memory devices. After finishing his doctoral work in 2002, he joined the faculty of Electrical and Computer Engineering, Cornell University, Ithaca, NY. His current research focuses on semiconductor optoelctronics and quantum optics.

Rajeev J. Ram was born in Hull, U.K., in 1970. He received the B.S. degree in applied physics from the California Institute of Technology, Pasadena, in 1991 and the M.S. and Ph.D. degrees in electrical engineering from the University of California, Santa Barbara (UCSB), in 1992 and 1996, respectively.

In 1997, he joined the faculty of Electrical Engineering, Massachusetts Institute of Technology, Cambridge. His current research interests are in semiconductor optoelectronics, quantum optics, and nanoscale electronics.

Dr. Ram is the recepient of the National Science Foundation Early Carrer Award and the Young Investigator Award of the Office of Naval Research.

Harmann A. Haus, deceased, was born in 1925 in Ljubljana, Slovenia. He received the B.Sc. degree from the Union College, Schenectady, NY, in 1949, the M.S. degree from the Rensselaer Polytechnic Institute, Troy, NY, in 1951, and the D.Sc. degree from the Massachusetts Institute of Technology (MIT), Cambridge, in 1954.

He joined the faculty of Electrical Engineering at MIT in 1954. He passed away in May 2003. He authored more than 350 journal articles and published six books.

Dr. Haus was a Fellow of the American Physical Society (APS) and the Optical Society of America (OSA) and a Member of the National Academy of Engineering, the National Academy of Sciences, and the American Academy of Arts and Sciences. He was the recepient of the 1984 Award of the IEEE Quantum Electronics and Applications Society, the 1987 Charles Townes Prize of the OSA, the 1990 Outstanding Paper Award of the IEEE TRANSACTIONS ON EDUCATION, the 1991 IEEE Education Medal, the 1994 Frederic Ives Medal of the OSA, the U.S. President's 1995 National Science Medal, the Ludwig-Wittgenstein-Preis Award of the Austrian Government in 1997, and the Willis E. Lamb Medal in 2001. He received honorary doctorates from the Union College, the Technical University of Vienna, and the University of Ghent.