ECE 5630 - Homework Assignment 1

January 31st, 2020

Due to: Thursday, February 13th, 2020 (at the beginning of the lecture)

Instructions: Submission in pairs is allowed. Prove and explain every step in your answers.

Errata: Fixed typo in the definition of $\kappa(\cdot|\cdot)$ in Question 7.

1) **Discrete probability spaces:** Let Ω be a countable set.
   a) Show that its power set 2^{Ω} is a σ-algebra.
   b) Let $p : \Omega \to [0,1]$ be a probability mass function (PMF), i.e., satisfying $\sum_{\omega\in\Omega} p(\omega) = 1$. Define a function $\mathbb{P}_p : 2^{\Omega} \to [0,1]$ by $\mathbb{P}_p(A) \triangleq \sum_{\omega\in A} p(\omega)$, for $A \in 2^{\Omega}$. Show that $(\Omega, 2^{\Omega}, \mathbb{P}_p)$ is a probability space.

2) **$\boldsymbol{\sigma}$-algebra**: Let Ω be an arbitrary set. All σ-algebras below are collections of subsets of Ω.
   a) Consider a sequence of σ-algebras $\{\mathcal{F}_n\}_{n=1}^{\infty}$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, for all $n \in \mathbb{N}$. Is $\mathcal{G} := \bigcup_{n=1}^{\infty} \mathcal{F}_n$ a σ-algebra? Either a proof (by showing it satisfies the required properties) or provide a counterexample.
   b) Let $\{\mathcal{F}_i\}_{i\in I}$ be an arbitrary (possible uncountable) collection of σ-algebras. Show that $\mathcal{H} := \bigcap_{i\in I} \mathcal{F}_i$ is a σ-algebra.
   c) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be arbitrary σ-algebras. Define their Cartesian product $\mathcal{F}_1 \times \mathcal{F}_2 := \{A_1 \times A_2 \mid A_1 \in \mathcal{F}_1,\ A_2 \in \mathcal{F}_2\}$, where $A_1 \times A_2$, the Cartesian product of two sets A_1 and A_2, is defined as $A_1 \times A_2 := \{(a_1, a_2) \mid a_1 \in A_1,\ a_2 \in A_2\}$. Either prove that $\mathcal{F}_1 \times \mathcal{F}_2$ is a σ-algebra or provide a counterexample.

3) **Properties of probability measures:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Prove the following properties of $\mathbb{P}$:
   a) Law of complement probability: $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$, $\forall A \in \mathcal{F}$, where $A^c = \Omega \setminus A$ is the complement of A.
   b) Monotonicity: If $A, B \in \mathcal{F}$ with $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$
   c) Union bound: For any $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, we have $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$
   d) Continuity of probability: Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of event increasing to $A \in \mathcal{F}$, i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $\bigcup_{n=1}^{\infty} A_n = A$. Similarly, let $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of event decreasing to $B \in \mathcal{F}$, i.e., $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} B_n = B$. Prove that:
      i) $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$. Deduce that for any $\{A'_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, we have $\lim_{m\to\infty} \mathbb{P}\left(\bigcup_{n=1}^{m} A'_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A'_n\right)$.
      ii) $\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B)$. Deduce that for any $\{B'_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, we have $\lim_{m\to\infty} \mathbb{P}\left(\bigcap_{n=1}^{m} B'_n\right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B'_n\right)$.
   e) Law of Total Probability: Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a partition of Ω (i.e., (i) $A_n \cap A_m = \emptyset$, $\forall n \neq m$; and (ii) $\bigcup_{n=1}^{\infty} A_n = \Omega$) and $B \in \mathcal{F}$. Then $\mathbb{P}(B) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)\mathbb{P}(B|A_n)$. Is your argument valid when $\mathbb{P}(A_{n'}) = 0$ for some $n' \in \mathbb{N}$?

4) **Measurability of indicators:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $A \in \mathcal{F}$, define the function $\mathbb{1}_A : \Omega \to \mathbb{R}$ by

$$\mathbb{1}_A(\omega) \triangleq \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases} .$$

   Prove that $\mathbb{1}_A$ is a random variable over $(\Omega, \mathcal{F}, \mathbb{P})$.

5) **Induced probability measure vs. probability law:** Let $X : \Omega \to \mathbb{R}^d$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with probability law $\mathbb{P}_X$, i.e., $\mathbb{P}_X(B) := \mathbb{P}\big(X^{-1}(B)\big)$, for $B \in \mathcal{B}(\mathbb{R}^d)$; $\mathbb{P}_X$ is a probability measure on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$.
   a) Assume X is a discrete random variable, i.e., $\mathrm{supp}(\mathbb{P}_X)$ is countable, and let $p_X : \mathrm{supp}(\mathbb{P}_X) \to [0,1]$ be its PMF, defined by $p_X(x) := \mathbb{P}_X(\{x\})$, $x \in \mathrm{supp}(\mathbb{P}_X)$. Prove that the probability measure $\mathbb{P}_{p_X}$ that is induced by the PMF p_X (see Question 1) coincides with the law of X, $\mathbb{P}_X$. In other words, show that $\mathbb{P}_{p_X}(B) = \mathbb{P}_X(B)$, for all $B \in \mathcal{B}(\mathbb{R}^d)$, where $\mathbb{P}_{p_X}(B) := \sum_{x \in \mathrm{supp}(\mathbb{P}_X) \cap B} p_X(x)$, for $B \in \mathcal{B}(\mathbb{R}^d)$, is the trivial extension of $\mathbb{P}_{p_X}$ from $\mathrm{supp}(\mathbb{P}_X)$ to the entire Borel σ-algebra on $\mathbb{R}^d$.

      Hint: Reduce your analysis from the continuous space $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$ to the discrete space $\big(\mathrm{supp}(\mathbb{P}_X), 2^{\mathrm{supp}(\mathbb{P}_X)}\big)$. Why is it suffices to consider only subsets of the form $B \cap \mathrm{supp}(\mathbb{P}_X)$, where $B \in \mathcal{B}(\mathbb{R}^d)$?

   b) Assume X is continuous with probability density function (PDF) $f_X : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, i.e., f_X is the derivative of the cumulative distribution function (CDF) F_X of X. Recall that any PDF f (a nonnegative integrable function with $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = 1$) induces a measure $\mathbb{P}_f$ on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$ by defining $\mathbb{P}_f(B) := \int_B f(x)\,\mathrm{d}x$, for all $B \in \mathcal{B}(\mathbb{R}^d)$. Prove that $\mathbb{P}_{f_X}$ coincides with $\mathbb{P}_X$ on the generating set of the Borel σ-algebra, i.e.,

$$\mathbb{P}_{f_X}\big((-\infty, a_1] \times \cdots \times (-\infty, a_d]\big) = \mathbb{P}_X\big((-\infty, a_1] \times \cdots \times (-\infty, a_d]\big), \quad \forall a_1, \ldots, a_d \in \mathbb{R}.$$

      Comment: In your proof, you may assume $d = 1$.

6) **Generation of samples with arbitrary distribution:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ be random variable on it. Denote the probability law of X by $\mathbb{P}_X$, and let $F_X : \mathbb{R} \to [0,1]$ be the CDF, i.e., $F_X(x) = \mathbb{P}_X\left((-\infty, x]\right)$, for all $x \in \mathbb{R}$.
   a) Show that $Y := F_X(X)$ is a random variable.
   b) Prove that Y is uniformly distributed on $[0,1]$. It suffices to show that F_Y, the CDF of Y, is given by $F_Y(y) = y$, for all $y \in [0,1]$, and then explain how the CDF argument extends to the entire probability law $\mathbb{P}_Y$.
   c) Let $U \sim \mathsf{Unif}[0,1]$ be a uniformly distributed random variable. Propose a measurable function $T : [0,1] \to \mathbb{R}$, such that $T(U)$ has distribution $\mathbb{P}_X$ (or, equivalently, CDF F_X). Prove your answer, but you are not require to establish measurability of T.

      Hint: The function T depends on F_X.

7) **Transition kernels:** Let $\Omega = \{0,1\}$ and consider the following construction of a transition kernel $\kappa(\cdot|\cdot)$ from $(\Omega, 2^\Omega)$ to itself. For $\omega \in \{0,1\}$, let $\mathsf{Ber}(\alpha_\omega)$ be a Bernoulli distribution with parameter $\alpha_\omega \in [0,1]$, i.e., as defined by the PMF $p_\omega(1) = 1 - p_\omega(0) = \alpha_\omega$. Set $\kappa(\cdot|0) = \mathsf{Ber}(\alpha_0)$ and $\kappa(\cdot|1) = \mathsf{Ber}(1-\alpha_1)$. This construction of $\kappa(\cdot|\cdot)$ is called the *binary channel with flip parameters* $(\alpha_0, \alpha_1) \in [0,1]^2$. Prove the following claims:
   i) $\kappa(\cdot|\cdot)$ is a transition kernel (verify the conditions in the definition).
   ii) Let $P_X = \mathsf{Ber}(0.5)$ and define $P_Y(B) := \mathbb{E}\big[\kappa(B|X)\big]$, for $B \in 2^\Omega$. Show that whenever $\alpha_0 = \alpha_1$, we have $P_Y = \mathsf{Ber}(0.5)$.
   iii) Give a counterexample to the symmetry of P_Y when $\alpha_0 \neq \alpha_1$.