ECE 5630 - Solutions to Homework Assignment 1

1) a) The statements $A \in 2^\Omega$ and $A \subset \Omega$ are equivalent. Clearly $\emptyset \in 2^\Omega$. If $A \in 2^\Omega$, i.e., $A \subseteq \Omega$, then $A^c = \Omega \setminus A \subset \Omega$ and thus $A^c \in 2^\Omega$. Finally, for $A_1, A_2, \ldots$ subsets of Ω consider the set $A = \bigcup_{n=1}^\infty A_n$. To show that $A \subseteq \Omega$ it suffices to show that for any $x \in A$ we have $x \in \Omega$. Let $x \in A$, by definition of countable unions we $x \in A_n$, for some $n \in \mathbb{N}$. Since $A_n \subset \Omega$ this implies $x \in \Omega$, as needed.

b) We need to show that $\mathbb{P}_p$ is a probability measure. Recall $\mathbb{P}_p(A) = \sum_{\omega \in A} p(\omega)$, for all $A \subseteq \Omega$. First

$$\mathbb{P}_p(\Omega) = \sum_{\omega \in \Omega} p(\omega) = 1.$$

Next, let $A_1, A_2, \ldots$ be disjoint subsets of Ω. Then:

$$\mathbb{P}_p\left(\bigcup_{i=1}^\infty A_n\right) = \sum_{a \in \cup_{n=1}^\infty A_n} p(\omega) = \sum_{n=1}^\infty \sum_{\omega \in A_n} p(\omega) = \sum_{n=1}^\infty \mathbb{P}_p(A_n),$$

where the second equality follows because $\{A_n\}$ are disjoint.

2) a) Consider $\Omega = \mathbb{N}$ and define $\mathcal{F}_n := \sigma\left(2^{\{1,\ldots,n\}}\right)$ as the σ-algebra (of subsets of Ω) generated by the power set of $\{1, \ldots, n\}$. This construction satisfies $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ by the inclusion of the generating sets. Let $\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{F}_n$ and consider the singleton sets $A_n := \{2n\}$. Clearly $A_n \in \mathcal{F}$ since $A_n \in \mathcal{F}_{2n}$. However, the countable union $A := \bigcup_{n=1}^\infty A_n = \{2, 4, 6, \ldots\}$ does not belong to any of the $\mathcal{F}_n$ σ-algebras, and therefore, $A \notin \mathcal{F}$. As σ-algebras are closed under countable unions, $\mathcal{F}$ cannot be a σ-algebra.

b) Since $\emptyset \in \mathcal{F}_i$, for all $i \in I$, we have $\emptyset \in \mathcal{H}$, by definition of intersection. Let $A \in \mathcal{H}$, i.e., $A \in \mathcal{F}_i$, for all $i \in I$, which implies $A^c \in \mathcal{F}_i$, for all $i \in I$ (since $\mathcal{F}_i$ are all σ-algebras), and thus $A \in \mathcal{H}$. Finally, if $A_1, A_2, \ldots \in \mathcal{H}$, we have $A_1, A_2, \ldots \in \mathcal{F}_i$, for all $i \in I$, and because $\mathcal{F}_i$ are σ-algebras we have $\bigcup_{k=1}^\infty A_k \in \mathcal{F}_i$, for all $i \in I$, which in turn gives $\bigcup_{k=1}^\infty A_k \in \mathcal{H}$, as desired.

c) No. Let $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{B}(\mathbb{R})$. The Cartesian product $\mathcal{F}_1 \times \mathcal{F}_2$ is not a σ-algebra on $\mathbb{R}^2$ since sets such as $(-\infty, a] \times (-\infty, b]$ are in $\mathcal{F}_1 \times \mathcal{F}_2$, but not their complements.

3) **Properties of probability measures:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Prove the following properties of $\mathbb{P}$:

a) $1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$, where the last equality is due to σ-additivity (noting that $A \cap A^c = \emptyset$).

b) Let $C = B \setminus A$, and so $B = A \cup C$ where A and C are disjoint. Then $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(C) \geq \mathbb{P}(A)$, where the last inequality uses non-negativity of probability.

c) Define $B_1 = A_1$ and $B_n = A_n \setminus \left\{\bigcup_{i=1}^{n-1} A_i\right\}$, for $n \geq 2$. By definition $\{B_n\}_{n=1}^\infty$ are disjoint and $B_n \subseteq A_n$, for all $n \in \mathbb{N}$. Noting that $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n$, we have

$$\mathbb{P}\left(\bigcup_{n=1}^\infty A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^\infty B_n\right) = \sum_{n=1}^\infty \mathbb{P}(B_n) \leq \sum_{n=1}^\infty \mathbb{P}(A_n),$$

where the 2nd equality is by σ-additivity and the 3rd equality is by monotonocity.

d) We prove only the claim for increasing events and union. The proof for decreasing event and intersection is similar.

Let $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$, for $n \geq 2$. Since $A_1 \subseteq A_2 \subseteq \dots$ then the B_n are disjoint. Also, $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i = A_n$, for all $n \in \mathbb{N}$. Similarly, $\bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty A_i = A$, by definition. Since the $\{B_n\}$ are disjoint we have

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{n=1}^\infty B_n\right) = \sum_{n=1}^\infty \mathbb{P}(B_n) = \lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}(B_i) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^n B_i\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

To see that this implies $\lim_{m\to\infty} \mathbb{P}\left(\bigcup_{n=1}^m A'_n\right) = \mathbb{P}\left(\bigcup_{n=1}^\infty A'_n\right)$, define $A_m = \bigcup_{n=1}^m A'_n$ (which is an increasing sequence of events), note that $\bigcup_{n=1}^\infty A'_n = \bigcup_{n=1}^\infty A_n$, and apply the previous claim

e) Consider the following:

$$\mathbb{P}(B) = \mathbb{P}\left(B \cap \left\{\bigcup_{n=1}^\infty A_n\right\}\right) = \mathbb{P}\left(\bigcup_{n=1}^\infty \{B \cap A_n\}\right) = \sum_{n=1}^\infty \mathbb{P}(B \cap A_n) = \sum_{n=1}^\infty \mathbb{P}(B)\mathbb{P}(B|A_n),$$

where the last equality follows by definition of conditional probability. The derivation still holds when $\mathbb{P}(A_{n'}) = 0$ for some $n' \in \mathbb{N}$ since then $\mathbb{P}(B \cap A_{n'}) = 0$ and we can remove this summand from the series.

4) We need to show that $\mathbb{1}_A^{-1}(B) \in \mathcal{F}$, for any $B \in \mathcal{B}(\mathbb{R})$. There are only 4 cases to considers:

- If $\{0,1\} \cap B = \emptyset$, then $\mathbb{1}_A^{-1}(B) = \emptyset \in \mathcal{F}$;
- If $\{0,1\} \cap B = \{1\}$, then $\mathbb{1}_A^{-1}(B) = A \in \mathcal{F}$;
- If $\{0,1\} \cap B = \{0\}$, then $\mathbb{1}_A^{-1}(B) = A^c \in \mathcal{F}$;
- If $\{0,1\} \cap B = \{0,1\}$, then $\mathbb{1}_A^{-1}(B) = \Omega \in \mathcal{F}$.

This concludes the proof.

5) a) It suffices to establish that $\mathbb{P}_X\big(B \cap \mathrm{supp}(\mathbb{P}_X)\big) = \mathbb{P}_{p_X}\big(B \cap \mathrm{supp}(\mathbb{P}_X)\big)$, for all $B \in \mathcal{B}(\mathbb{R}^d)$, since both measures nullify outside of $\mathrm{supp}(\mathbb{P}_X)$. Thus, let $B \in 2^{\mathrm{supp}(\mathbb{P}_X)}$ and because $\mathrm{supp}(\mathbb{P}_X)$ is discrete, we can represent $B = \bigcup_{n=1}^\infty \{b_n\}$ as a countable union of singletons. We have

$$\mathbb{P}_{p_X}(B) = \sum_{b\in B} p_X(b) = \sum_{n=1}^\infty \mathbb{P}_X(\{b_n\}) = \mathbb{P}_X\left(\bigcup_{n=1}^\infty \{b_n\}\right) = \mathbb{P}_X(B).$$

b) Let $A = (-\infty, a_1] \times \cdots \times (\infty, a_d]$ and consider

$$\mathbb{P}_{f_X}(A) = \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_d} f_X(x_1, \dots, x_d)\, \mathrm{d}x_1 \cdots \mathrm{d}x_d = F_X(A) = \mathbb{P}_X(A),$$

where the last equality is since the CDF F_X is the restriction of the law $\mathbb{P}_X$ to the generating set of the Borel σ-algebra.

6) a) Note that Y is a mapping from $\Omega = \mathbb{R}$ to $\mathcal{Y} = [0,1]$. Endow each space with the its Borel σ-algebra (for $\mathcal{Y}$ the Borel σ-algebra is generated by, e.g., $\{(0,a]\}_{a\in(0,1]}$). We first verify measurability of generating sets: Fix $B \in \mathcal{B}_a = \big(0, a\big]$ and note that $Y^{-1}(B_a) = \big(0, F_X^{-1}(a)\big]$, where $F_X^{-1} := \inf\big\{x \in \mathbb{R} : a \leq F_X(x)\big\}$ is the generalized inverse (also known as quantile function) of F_X. Clearly $Y^{-1}(B_a) \in \mathcal{B}(\mathbb{R})$. This, in turn, implies measurability of any $B \in \mathcal{B}\big((0,1]\big)$, since Borel sets can be represented through a countable number of unions, intersections and complements, all of which commute with the inverse map.

b) For the CDF of Y, we have

$$F_Y(y) = \mathbb{P}_Y\big((-\infty, y]\big) = \mathbb{P}_X\left(Y^{-1}\big((-\infty, y]\big)\right) = \mathbb{P}_X\left(\big(-\infty, F_X^{-1}(y)\big]\right) = F_X\left(F_X^{-1}(y)\right) = y.$$

The extension of F_Y above to the uniform probability measure on $[0,1]$ follows from Carathéodory's extension theorem.

c) This holds for $T(u) = F_X^{-1}(u)$, where F_X^{-1} is the generalized inverse defined above. This is known as the inverse transform sampling method. Denoting the law of U by $\mathbb{P}_U$, we have $F_T(t) = \mathbb{P}_U\Big(\big[0, F_X(t)\big]\Big) = F_X(t)$, i.e., T and X are equal in distribution (since a CDF uniquely defines the probability law).

7) i) We begin by noting that $\kappa(\cdot|\omega)$, for any $\omega \in \Omega$, is a Bernoulli measure, as the first condition requires. Next, fixing $A \subseteq \Omega$, we need to show that $\kappa(A|\cdot) : \Omega \to \mathbb{R}$ is a measurable function (random variable). This is trivial since any function is measurable with respect to the power set σ-algebra, which establishes $\kappa(\cdot|\cdot)$ as a valid transition kernel. Still, as a further clarification, notice that there are 4 functions to consider in the second step, namely,

- $\kappa_\emptyset(\cdot) := \kappa(\emptyset|\cdot)$, which equals 0 for all inputs, i.e., $\kappa_\emptyset(0) = \kappa_\emptyset(1) = 0$;
- $\kappa_\Omega(\cdot) := \kappa(\Omega|\cdot)$, which equals 1 for all inputs, i.e., $\kappa_\emptyset(0) = \kappa_\emptyset(1) = 1$;
- $\kappa_0(\cdot) := \kappa(\{0\}|\cdot)$, which equals $\kappa_0(0) = 1 - \alpha_0$ and $\kappa_0(1) = \alpha_1$ (this is not a probability measure, but is a measurable function);
- $\kappa_1(\cdot) := \kappa(\{1\}|\cdot)$, which equals $\kappa_0(1) = \alpha_0$ and $\kappa_1(1) = 1 - \alpha_1$.

ii) Since $\mathrm{supp}(P_Y) = \{0,1\}$ it is enough to show that $P_Y\big(\{0\}\big) = 0.5$. Denote $\alpha = \alpha_0 = \alpha_1$ and consider:

$$P_Y\big(\{0\}\big) = \mathbb{E}\big[\kappa\big(\{0\}\big|X\big)\big] = \frac{1}{2}\kappa\big(\{0\}\big|0\big) + \frac{1}{2}\kappa\big(\{0\}\big|1\big) = \frac{1}{2}(1-\alpha) + \frac{1}{2}\alpha = \frac{1}{2}.$$

iii) Let $\alpha_0 = 0$ and $\alpha_1 = \frac{1}{2}$. Repeating the above calculation gives

$$P_Y\big(\{0\}\big) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$