Lecture 7: Data Processing Inequality

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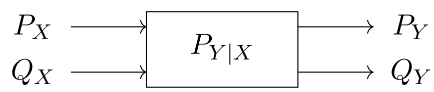
We will introduce the data processing inequality (DPI) here. In essence, the DPI shows that the f-divergence between two distributions does not decrease when we push it through a transition kernel. This can be thought of as follows: pushing two observations X and Y through a channel will only make it harder to distinguish between them.



Theorem 1 (Data Processing Inequality) *Let $P_X, Q_X \in \mathcal{P}(\mathcal{X})$ and $P_{Y|X}$ be a transition kernel from $(\mathcal{X}, \mathcal{F})$ to $(\mathcal{Y}, \mathcal{G})$. Let $P_Y, Q_Y \in \mathcal{P}(\mathcal{Y})$ be the transformation of P_X and Q_X, respectively, when pushed through $P_{Y|X}$, i.e., $P_X(B) = \int_{\mathcal{X}} P_{Y|X}(B \mid x) \mathsf{d}P_X(x)$. Then, for any f-divergence, we have that*

$$D_f(P_X \| Q_X) \geq D_f(P_Y \| Q_Y).$$

Example 7.1

1. <u>Gaussian Convolutions:</u> *Let $X \sim P_X$, $X' \sim Q_X$, and $Z, Z' \sim \mathcal{N}_\sigma := \mathcal{N}(0, \sigma^2 \mathrm{I}_d)$ be independent random variables. Define $Y := X + Z$ and $Y' := X' + Z'$. Here, the transition kernel is $P_{Y|X}(\cdot \mid x) = \mathcal{N}(x, \sigma^2 \mathrm{I}_d)$.*

   Recall that for two independent random variables $W \sim \mu$ and $W' \sim \nu$, it holds that $W + W' \sim \mu * \nu$ where $\mu * \nu$ is the convolution of μ and ν defined as

   $$(\mu * \nu)(A) = \int_{\mathcal{X}} \int_{\mathcal{Y}} \mathbb{1}_{\{x+y \in A\}} \mathsf{d}\mu(x) \mathsf{d}\nu(y),$$

   for any measurable $A \subseteq \mathcal{X} + \mathcal{Y} := \{x + y : x \in \mathcal{X},\ y \in \mathcal{Y}\}$ (note that if $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, then $\mu * \nu \in \mathcal{P}(\mathcal{X} + \mathcal{Y})$).

   It follows that $Y \sim P_X * \mathcal{N}_\sigma$ and $Y' \sim Q_X * \mathcal{N}_\sigma$. The DPI implies

   $$D_f(P_X \| Q_X) \geq D_f(P_X * \mathcal{N}_\sigma \| Q_X * \mathcal{N}_\sigma).$$

2. <u>Deterministic Functions:</u> *Let $X \sim P_X$, $X' \sim Q_X$ and set $Y = g(X)$, $Y' = g(X')$ for a deterministic measurable g. The transition kernel is $P_{Y|X}(\cdot \mid x) = \delta_{g(x)}(\cdot)$, where δ_a is the Dirac measure centered at $a \in \mathcal{X}$, i.e., $\delta_a(A) = \mathbb{1}_A(a)$, for any A measurable.*

   (i) *Let E be any measurable event, define $g(x) = \mathbb{1}_{\{x \in E\}}$ and set $Y = g(X)$. Note that Y is a binary random variable with $P_Y(\{1\}) = P_X(E)$. This implies that $Y = \mathbb{1}_{\{X \in E\}} \sim \mathrm{Ber}(P_X(E))$ and $Y' = \mathbb{1}_{\{X' \in E\}} \sim \mathrm{Ber}(Q_X(E))$. By the data processing inequality, we obtain*

   $$D_f(P_X \| Q_X) \geq D_f\Big(\mathrm{Ber}\big(P_X(E)\big) \Big\| \mathrm{Ber}\big(Q_X(E)\big)\Big),$$

   for all measurable E.

(ii) Consider $g(x_1, x_2) = x_1$, *and let* $X = (X_1, X_2) \sim P_{X_1,X_2}$, $X' = (X_1', X_2') \sim Q_{X_1,X_2}$, $Y = g(X_1, X_2) = X_1$, *and* $Y' = g(X_1', X_2') = X_1'$. *It follows that* $Y \sim P_{X_1}$ *and* $Y \sim Q_{X_1}$. *Applying the data processing inequality gives*

$$D_f(P_{X_1,X_2} \| Q_{X_1,X_2}) \geq D_f(P_{X_1} \| Q_{X_1})$$

By Item (iv) from the properties of f*-divergences we have that if* $P_{X_2|X_1} = Q_{X_2|X_1}$ *then equality above holds.*

Proof of DPI: Throughout this proof we use the shorthand $\frac{\mathsf{d}P}{\mathsf{d}Q} = \frac{\mathsf{d}P/\mathsf{d}\lambda}{\mathsf{d}Q/\mathsf{d}\lambda}$, where λ is a measure that dominates both P and Q (e.g., $\lambda = P + Q$), and $\mathsf{d}P/\mathsf{d}\lambda$ is the Radon-Nikodym derivative of P w.r.t. λ.

First, recall that if $P_{XY} = P_X P_{Y|X}$ and $Q_{XY} = Q_X P_{Y|X}$, then

$$D_f(P_X \parallel Q_X) = D_f(P_{XY} \parallel Q_{XY}) = \mathbb{E}_{Q_{XY}}\left[f\left(\frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\right)\right].$$

Using the law of total expectation, we get

$$\mathbb{E}_{Q_{XY}}\left[f\left(\frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\right)\right] = \mathbb{E}_{Q_Y}\left[\mathbb{E}_{Q_{X|Y}}\left[f\left(\frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\right)\middle|Y\right]\right].$$

As f is convex, applying Jensen's inequality yields

$$\mathbb{E}_{Q_Y}\left[\mathbb{E}_{Q_{X|Y}}\left[f\left(\frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\right)\middle|Y\right]\right] \geq \mathbb{E}_{Q_Y}\left[f\left(\mathbb{E}_{Q_{X|Y}}\left[\frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\middle|Y\right]\right)\right].$$

To conclude the proof, it suffices to show that

$$\mathbb{E}_{Q_{X|Y}}\left[\frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\middle|Y\right] = \frac{\mathsf{d}P_Y}{\mathsf{d}Q_Y}.$$

It holds that

$$\mathbb{E}_{Q_{X|Y}}\left[\frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\middle|Y\right] = \int_{\mathcal{X}} \frac{\mathsf{d}P_{XY}}{\mathsf{d}Q_{XY}}\mathsf{d}Q_{X|Y} = \int_{\mathcal{X}} \frac{\mathsf{d}P_Y \mathsf{d}P_{X|Y}}{\mathsf{d}Q_Y \mathsf{d}Q_{X|Y}}\mathsf{d}Q_{X|Y} = \int_{\mathcal{X}} \frac{\mathsf{d}P_Y}{\mathsf{d}Q_Y}\mathsf{d}P_{X|Y} = \frac{\mathsf{d}P_Y}{\mathsf{d}Q_Y}.$$

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