ECE 6970 - Homework Assignment 1

September 13th 2019

Due to: Thursday, September 26th, 2019 (at the beginning of the lecture)

Instructions: Submission in pairs is allowed. Prove and explain every step in your answers.

Errata: Fixes to Questions 2(e) and 7(b) were introduced. Thanks to Kia Khezeli.

1) **Discrete probability spaces:** Let $\mathcal{X}$ be a countable set.
   a) Show that its power set $2^{\mathcal{X}}$ is a valid σ-algebra.
   b) Let $p : \mathcal{X} \to [0,1]$ be a probability mass function (PMF), i.e., satisfying $\sum_{x\in\mathcal{X}} p(x) = 1$. Define a function $\pi_p : 2^{\mathcal{X}} \to [0,1]$ by $\pi_p(A) \triangleq \sum_{x\in A} p(x)$, for $A \in 2^{\mathcal{X}}$. Show that $(\mathcal{X}, 2^{\mathcal{X}}, \pi_p)$ is a probability space.

2) **Properties of probability measures:** Let $(X, \mathcal{F}, \pi)$ be a probability space. Prove the following properties of π:
   a) Law of complement probability: $\pi(A) = 1 - \pi(A^c)$, $\forall A \in \mathcal{F}$, where $A^c = \mathcal{X} \setminus A$ is the complement of A.
   b) Monotonicity: If $A, B \in \mathcal{F}$ with $A \subseteq B$, then $\pi(A) \leq \pi(B)$
   c) Union bound: For any $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, we have $\pi\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \pi(A_n)$
   d) Continuity of probability: Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of event increasing to $A \in \mathcal{F}$, i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and $\bigcup_{n=1}^{\infty} A_n = A$. Similarly, let $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of event decreasing to $B \in \mathcal{F}$, i.e., $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ and $\bigcap_{n=1}^{\infty} B_n = B$. Prove that:
      i) $\lim_{n\to\infty} \pi(A_n) = \pi(A)$. Deduce that for any $\{A'_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, we have $\lim_{m\to\infty} \pi\left(\bigcup_{n=1}^{m} A_n\right) = \pi\left(\bigcup_{n=1}^{\infty} A_n\right)$.
      ii) $\lim_{n\to\infty} \pi(B_n) = \pi(B)$. Deduce that for any $\{B'_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, we have $\lim_{m\to\infty} \pi\left(\bigcap_{n=1}^{m} B_n\right) = \pi\left(\bigcap_{n=1}^{\infty} B_n\right)$.
   e) Law of total probability: Prove that for any partition $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ of $\mathcal{X}$ (i.e., (i) $A_n \cap A_m = \emptyset$, $\forall n \neq m$; and (ii) $\bigcup_{n=1}^{\infty} A_n = \mathcal{X}$) and $B \in \mathcal{F}$, we have $\pi(B) = \sum_{n=1}^{\infty} \pi(A_n)\pi(B|A_n)$. Is your argument valid when $\pi(A_{n'}) = 0$ for some $n' \in \mathbb{N}$?

3) **Measurability of indicators:** Let $(X, \mathcal{F}, \pi)$ be a probability space. For $A \in \mathcal{F}$, define the function $\mathbb{1}_A : \mathcal{X} \to \mathbb{R}$ by

$$\mathbb{1}_A(x) \triangleq \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

   a) Prove that $\mathbb{1}_A$ is a random variable over $(X, \mathcal{F}, \pi)$.
   b) Show that $\mathbb{1}_A(x) = \delta_x(A)$ for every $A \in \mathcal{F}$ and $x \in \mathcal{X}$, where δ_x is the Dirac measure centered on x. Despite the above equality, explain the difference between $\mathbb{1}_A$ and δ_x.

4) **Composition of transport maps:** Let $\mu \in \mathcal{P}(\mathcal{X})$, $\nu \in \mathcal{P}(\mathcal{Y})$ and $\eta \in \mathcal{P}(\mathcal{Z})$ be probability measures. Suppose that $T : \mathcal{X} \to \mathcal{Y}$ transports μ to ν and that $S : \mathcal{Y} \to \mathcal{Z}$ transports ν to η, i.e., we have $T_{\#}\mu = \nu$ and $S_{\#}\nu = \eta$. Prove that $S \circ T$ transports μ to η (rederive any relevant steps shown in class and in particular show that $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$).

5) **Nonexistence of transport maps:** Find probability measures $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$ for which no $T : \mathcal{X} \to \mathcal{Y}$ with $T_{\#}\mu = \nu$ exists, where $\big|\operatorname{supp}(\mu)\big|, \big|\operatorname{supp}(\nu)\big| < \infty$ and:
   a) $\big|\operatorname{supp}(\mu)\big| < \big|\operatorname{supp}(\nu)\big|$;
   b) $\big|\operatorname{supp}(\mu)\big| = \big|\operatorname{supp}(\nu)\big|$;
   c) $\big|\operatorname{supp}(\mu)\big| > \big|\operatorname{supp}(\nu)\big|$.

6) **Lemma for weak convergence to null:** In class we showed that the set of transport maps between two given probability measure might not be closed with respect to weak convergence of functions in $L^1\big([0,1)\big)$ endowed with the uniform measure. Our proof relied on the following lemma.

   Lemma *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1\big([0,1)\big)$. A sufficient condition for $f_n \rightharpoonup 0$ as $n \to \infty$ is:*
   (i) $\big\|f_n\big\|_{L^1\left([0,1)\right)}$ is uniformly bounded; and
   (ii) For any $(a,b) \subseteq [0,1)$, we have $\int_a^b f_n(x)dx \to 0$ as $n \to \infty$.

   Prove this lemma. You may rely on the fact that for any $g \in L^{\infty}\big([0,1)\big)$ and $\epsilon > 0$ there exists a simple function $\sum_{i=1}^{m} c_i \mathbb{1}_{(a_i,b_i)}$, such that $\big\|g - \sum_{i=1}^{m} c_i \mathbb{1}_{(a_i,b_i)}\big\|_{L^{\infty}([0,1))} < \epsilon$.

7) **Coupling Gaussian measures:** Let $\mu = \mathcal{N}(0,\Sigma_1)$ and $\nu = \mathcal{N}(0,\Sigma_2)$ be two d-dimensional centered Gaussian measures with (nonsingular) covariance matrices Σ_1 and Σ_2, respectively.
   a) Write out the probability density function for the product coupling $\pi = \mu \times \nu$.
   b) Let $X \sim \mu$, $Y \sim \nu$ and assume $\Sigma_2 \succcurlyeq \Sigma_1$ in the positive semi-definite sense (i.e., $a^{\top}(\Sigma_2 - \Sigma_1)a \geq 0,\ \forall a \in \mathbb{R}^d$). Further let $Z \sim \mathcal{N}(0, \Sigma_2 - \Sigma_1)$ be independent of X and define a new pair of random variables (X', Y') by $X' = X$ and $Y' = X + Z$. Prove that (X', Y') is a coupling of $X \sim \mu$ and $Y \sim \nu$. Write out the join probability density function of (X', Y').
   c) Assume $d = 1$ and $X, Y \sim \mathcal{N}(0,\sigma^2)$, for $\sigma > 0$ (i.e., X and Y are now identically distributed). Propose <u>three different</u> couplings of X and Y such that their joint law in <u>not</u> Gaussian. Prove your claims.