Information Storage Capacity of Interacting Particle Systems

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Beyond IID in Information Theory 8

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Storing Information Inside Matter
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1. Writing data
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Goal: Study information storage capacity while:
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- Distilling notion of storage from particular technology
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- Capturing interparticle interaction and system’s dynamics
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**Goal:** Study information storage capacity while:

- Distilling notion of storage from particular technology
- Capturing interparticle interaction and system’s dynamics
- How much data can be stored and for how long?
Operational Framework
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Stochastic Ising Model:
Operational Framework

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\pi_\beta(\sigma) \propto e^{-\beta H(\sigma)}
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Operational Framework

\[ m \xrightarrow{X_0} \text{Enc} \xrightarrow{t \text{ time system dynamics}} X_t \xrightarrow{\hat{m}} \text{Dec} \]

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**Warm** \((\beta \text{ small})\) ➞ Weak interactions

**Cold** \((\beta \text{ large})\) ➞ Strong interactions
Measuring Information Storage

$\begin{align*}
\text{Enc} & \quad \xrightarrow{m} \quad X_0 \\
& \quad \xrightarrow{t \text{ steps of Galuber dynamics}} \\
& \quad \xrightarrow{X_t} \quad \text{Dec} \\
& \quad \xrightarrow{\hat{m}}
\end{align*}$
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- **Joint distribution:** \((X_0, X_t) \sim P_{X_0} P^t, \quad P - \text{transition kernel.}\)
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**Warm**: \(n\)-fold DM BSC \(\left(\frac{1}{2} + o(1)\right)\) after \(t = O(n)\).
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**Cold:** Can interactions (memory) help?
Zero-Temperature Dynamics \( (\beta \to \infty) \)

**Majority Update:**
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**Q1:** What (if anything) can be stored for infinite time?
Storing for Infinite Time

Theorem (G.-Bresler-Polyanskiy’19)

For the zero-temp. SIM on $\sqrt{n} \times \sqrt{n}$ grid \[ I_n(\infty) := \lim_{t \to \infty} I_n(t) = \Theta(\sqrt{n}) \]
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Achievability

- **Stable Configurations:** $\sigma \in \Omega$ is stable if $P(\sigma, \sigma) = 1$ (ground states).
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- All 2-striped config. are stable.
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[Diagram of the grid]
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Converse:
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$\implies \lim_{t \to \infty} P\left(X_t \in \{\text{Stripes}\}\right) = 1$
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$$\implies \lim_{t \to \infty} \mathbb{P}\left(X_t \in \{\text{Stripes}\}\right) = 1 \implies I_n(\infty) = O(\sqrt{n})$$
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Q2: Can we do better than $\sqrt{n}$ for finite superlinear $t$?
Storing $\omega(\sqrt{n})$ Bits for Superlinear Time

**Theorem (G.-Bresler-Polyanskiy’19)**

Let $a(n) = o(n)$. Then $\exists c > 0$ s.t. $I_n(t) = \Omega\left(\frac{n}{a(n)}\right)$, $\forall t \leq c \cdot a(n) \cdot n$. 
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**Codebook Construction:**
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- Tile grid with mono. subsquares of side $\sqrt{a(n)}$
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- $X_0 \sim \text{Unif}(C)$, $C \triangleq \{\sigma \text{ with this structure}\}$
Storing $\omega(\sqrt{n})$ Bits for Superlinear Time

**Theorem (G.-Bresler-Polyanskiy’19)**

Let $a(n) = o(n)$. Then $\exists c > 0$ s.t. $I_n(t) = \Omega\left(\frac{n}{a(n)}\right)$, $\forall t \leq c \cdot a(n) \cdot n$.

**Codebook Construction:**

- Tile grid with mono. subsquares of side $\sqrt{a(n)}$
- Separate by all-minus 2-strips
- $X_0 \sim \text{Unif}(C), \; C \triangleq \{ \sigma \text{ with this structure} \}$
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**Subsquare:**

![Subsquare Diagram]
Storing $\omega(\sqrt{n})$ Bits for Superlinear Time

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---

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- **Flip Probability:** $q_t = \mathbb{P}(\tau \leq t)$, $\tau = \inf \left\{ t : X_0^{(c)} = \square & X_t^{(c)} = \blacksquare \right\}$. 
Storing $\omega(\sqrt{n})$ Bits for Superlinear Time

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## Storage in Zero-Temp. SIM - Summary

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Instability of Zero-Temp. SIM

Grid with External Field:
Instability of Zero-Temp. SIM

Grid with External Field:

- **Hamiltonian:** \( \mathcal{H}(\sigma) = -\left( \sum_{\{u,v\} \in \mathcal{E}} \sigma(u)\sigma(v) + \hbar \sum_{v \in \mathcal{V}} \sigma(v) \right) \)
Instability of Zero-Temp. SIM

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For zero-temp. SIM on $\sqrt{n} \times \sqrt{n}$ grid with external field: $I_n(t) = \Theta(n)$, $\forall t$
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For zero-temp. SIM on Honeycomb lattice with $n$ vertices: $I_n(t) = \Theta(n)$, $\forall t$
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**Theorem (G.-Bresler-Polyanskiy’19)**

Fix $\epsilon \in \left( 0, \frac{1}{2} \right)$, $\gamma > 0$. For $\beta$ sufficiently large there exist $c > 0$ s.t.

$$I(X_0; X_t) \leq \log 2 + \epsilon_n(\beta),$$

for all $t \geq n \cdot e^{c\beta n^{1/4} + \epsilon}$, where $X_0 \sim \pi$ and $\lim_{n \to \infty} \epsilon_n(\beta) = 0$. 

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- $\{X^\sigma_t\}_t$ is the chain initiated at $\sigma \in \Omega_n$
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Proposition (Martinelli’94)

Let \( \epsilon, \gamma \) be as before. For \( \beta \) sufficiently large there exist \( c > 0 \) s.t.

\[
\sum_{\sigma \in \Omega_n: m(\sigma) > 0} \pi(\sigma) \mathbb{P}(X_t^\sigma \neq X_t^\Box) \leq e^{-\gamma \sqrt{n}}, \quad \forall t \geq n \cdot e^{c\beta n^{\frac{1}{4}} + \epsilon}
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Reduction to Single Stripe Analysis

- **Denote:** \( t_f \triangleq e^{c\beta} \); \( X_t^{(j)} \triangleq X_t^{(j)} \big|_{\text{Stripe } j} \); \( X_t^{[j]} \triangleq (X_t^{(k)})_{k=1}^j \)
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\[
I_n^{(\beta)}(t) \geq \sum_j I(X_0^{(j)}; X_{t_f} \big| X_0^{[j-1]}) \\
\geq \sum_j I(X_0^{(j)}; \psi_j(X_{t_f}) \big| X_0^{[j-1]}) \\
\geq \Theta(\sqrt{n}) \cdot C_{\text{BSC}}\left( P(\text{More than half stripe flipped}) \right)
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Reduction to Single Stripe Analysis

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\( \Rightarrow \) Suffices to analyze \( \mathbb{P}(\text{More than half stripe flipped}) \)
Single Stripe Case: Main Result

Bottom 1-Stripe:
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*Fix any $c, C \in (0, 1)$. For $\beta$ and $n$ sufficiently large, we have*

$$\mathbb{E}N(+) (t) \geq C \sqrt{n}, \quad \forall t \leq e^{c\beta}.$$
Pluses may spread out above bottom stripe
Single Stripe Case: Challenges & Solutions

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Single Stripe Case: Challenges & Solutions

⚠️ Pluses may spread out above bottom stripe

**Fix:** Prohibit minus-spins from flipping (speedup)

⚠️ **Interleaved Dynamics:** 2 types of flips

► **Sprinkle:** Flip w/ all-plus horizontal neighbors

► **Erosion:** Flip w/ at least one minus horizontal neighbor

**Expected Behavior:**

1. Initially chain stays close to $X_0$ w/ occasional sprinkles
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1. Initially chain stays close to $X_0$ w/ occasional sprinkles
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$\implies$ Dominate $\{X_t\}_t$ by a phase-separated dynamics
Single Stripe Case: Phase-Separated Dynamics (1)

- Consider continuous-time dynamics (i.i.d. Poisson clocks at each site)
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- $0 < t_1 < \ldots < t_k < t_f$ are the $k$ clock rings (at $v_1, \ldots, v_k$) until $t_f$
- Define new dynamics $\{\tilde{X}_t\}_{t \in [0, 2t_f]}$ with first $2k$ clock rings and flips

\[
\tau_j = \begin{cases} 
    t_j, & j \in [k] \\
    t_{j-k} + t_f, & j \in [k+1 : 2k]
\end{cases}, \quad u_j = \begin{cases} 
    v_j, & j \in [k] \\
    v_{j-k}, & j \in [k+1 : 2k]
\end{cases}
\]
Consider continuous-time dynamics (i.i.d. Poisson clocks at each site)

0 < t_1 < \ldots < t_k < t_f are the k clock rings (at v_1, \ldots, v_k) until t_f

Define new dynamics \{\tilde{X}_t\}_{t \in [0, 2t_f]} with first 2k clock rings and flips

\[ \tau_j = \begin{cases} 
t_j, & j \in [k] 
t_j - k + t_f, & j \in [k + 1 : 2k] 
\end{cases}, \quad \upsilon_j = \begin{cases} 
v_j, & j \in [k] 
v_j - k, & j \in [k + 1 : 2k] 
\end{cases} \]

\[ \{X_t\}_{t \in [0, t_f]} \]
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Single Stripe Case: Phase-Separated Dynamics (2)

\[ \{ \tilde{X}_t \}_{t \in [0,2t_f]} \]
Single Stripe Case: Phase-Separated Dynamics (2)

\( \{ \tilde{X}_t \}_{t \in [0,2t_f]} \)

Blocking Rule:
Single Stripe Case: Phase-Separated Dynamics (2)

\[ \{ \tilde{X}_t \}_{t \in [0,2t_f]} \]

Blocking Rule:

1. For \( t < t_f \) allow only sprinkle flips (wrt original \( \{ X_t \}_{t \in [0,t_f]} \) )
Single Stripe Case: Phase-Separated Dynamics (2)

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* Adjust Poisson clock rates of \( \{\tilde{X}_t\}_{t \in [0,2t_f]} \) to neighborhoods
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Observations:
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Observations:

- Erosion flips in \( \{X_t\}_{t \in [0,t_f]} \) \( \implies \) Erosion flips in \( \{\tilde{X}_t\}_{t \in [t_f,2t_f]} \)
**Blocking Rule:**

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**Observations:**

- Erosion flips in \( \{X_t\}_{t \in [0,t_f]} \) ⟷ Erosion flips in \( \{\tilde{X}_t\}_{t \in [t_f,2t_f]} \)
- Erosion flip rates in \( \{\tilde{X}_t\}_{t \in [t_f,2t_f]} \) are faster.
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\[ \{ \tilde{X}_t \}_{t \in [0,2t_f]} \]

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⋆ Adjust Poisson clock rates of \( \{ \tilde{X}_t \}_{t \in [0,2t_f]} \) to neighborhoods

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- Erosion flip rates in \( \{ \tilde{X}_t \}_{t \in [t_f,2t_f]} \) are faster.

\[ \implies \text{New dynamics is a speedup:} \quad \mathbb{E}N^{(+)}(t_f) \geq \mathbb{E}\tilde{N}^{(+)}(2t_f) \]
Single Stripe Case: Phase-Separated Dynamics (3)

Sprinkle Analysis $[0, t_f]$: Ends w/ runs of ‘+’s separated by ‘-’ sprinkles
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Q: What is the typical length of a run (contig) & how many of them?
**Single Stripe Case: Phase-Separated Dynamics (3)**

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- Approx. bottom stripe sites by i.i.d. \( \text{Exp}(p_\beta) \), \( p_\beta \triangleq P(\text{Sprinkle}) \)
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- Show $\mathbb{E}[\text{Number of contigs of this length}] \gtrsim \frac{\sqrt{n}}{2-p_\beta}$
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Erosion Analysis $(t_f, 2t_f)$: Contig eaten w/ speed $\phi_\beta \triangleq \frac{e^\beta}{e^{\beta} + e^{-\beta}}$ (2 sides)
Single Stripe Case: Phase-Separated Dynamics (3)

Sprinkle Analysis \([0, t_f]\): Ends w/ runs of ‘+’s separated by ‘-’ sprinkles

\[\begin{array}{cccccccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline \\
\end{array}\]

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Erosion Analysis \((t_f, 2t_f]\): Contig eaten w/ speed \(\phi_\beta \triangleq \frac{e^\beta}{e^\beta + e^{-\beta}} (2\text{ sides})\)

- \(\left\{\text{Half contig eaten in } t_f \text{ time}\right\} = \left\{\sum_{i=1}^{\ell_\beta/2} \text{Exp}(\phi_\beta) \leq t_f \right\}\)
**Single Stripe Case: Phase-Separated Dynamics (3)**

**Sprinkle Analysis \([0, t_f]::** Ends w/ runs of ‘+’s separated by ‘-’ sprinkles

![Diagram showing sprinkles and runs]

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**Erosion Analysis \((t_f, 2t_f)::** Contig eaten w/ speed \(\Phi_\beta \triangleq \frac{e^\beta}{e^\beta + e^{-\beta}}\) (2 sides)

- Half contig eaten in \(t_f\) time \[\{\sum_{i=1}^{\ell_\beta/2} \text{Exp}(\Phi_\beta) \leq t_f\}\]
- Show latter probability is small and conclude proof