Discrete Probability and Randomized Algorithms
Assignment One
Due: 9/19

It is ok to discuss the problems when stuck, but not ok to simply search for solutions. Please clearly explain the problems where you received any help from any source other than yourself or the instructor. There is NO penalty for this, but it will help us structure future assignments.

Problem 1. (Randomness generation). Throughout this course we will assume that we have access to arbitrary amount of randomness, namely we can say statements like "choose a number uniformly at random from the set S", etc. This problem discusses a few points about the process of generating randomness.

1. Suppose you have a coin with $\Pr(H) = p$. You toss the coin independently to generate tosses $X_1, X_2, \ldots$. Let T be the first time such that $X_{T-1} = H$, and $X_T = T$. Show that $\mathbb{E}[T] \le \frac{10}{p(1-p)}$.

   In the next two parts (2,3) you should design Las Vegas algorithms, which may never terminate.

2. Given a coin with an unknown $\Pr(H)$ that you can flip as many times as you wish. Describe a scheme to generate a random variable Y such that $\Pr(Y = H) = \Pr(Y = T) = 0.5$. Can you ensure that the **expected** number of tosses used by the scheme is at most $\frac{100}{p(1-p)}$.

3. Given a fair coin, and a number $p \in [0, 1]$. Generate a random variable Y such that $\Pr(Y = H) = p$, and $\Pr(Y = T) = 1 - p$ by flipping the coin such that the expected number of tosses is at most 100.

   The next part shows that assuming Las Vegas algorithms is essential.

4. Show that if p is irrational, any algorithm solves part 3 cannot have a bounded worst case running time.

Problem 2. (Fixed points and cycles). Let $[n] := \{1, \ldots, n\}$, and let $\sigma : [n] \to [n]$ be a permutation of $[n]$. A number i is a fixed point of σ if $\sigma(i) = i$. Let $f(\sigma)$ be the number of fixed points of σ. The cycle corresponding to a number i is the set $C(i) := \{i, \sigma(i), \sigma(\sigma(i)), \ldots\}$. For example, consider the permutation $\sigma = 5\ 6\ 3\ 2\ 1\ 4$. Then, the only fixed point is 3, and $C(1) = C(5) = \{1, 5\}$, $C(3) = \{3\}$, and $C(2) = C(4) = C(6) = \{2, 4, 6\}$.

1. Suppose σ is uniformly chosen from all possible $n!$ permutations. Show that $\mathbb{E}[f(\sigma)] = 1$.

2. Suppose f_k denotes the number of permutations of $[n]$ with exactly k fixed points. Then show that $\sum_{k=0}^{n} k \cdot f_k = n!$.

3. Suppose σ is uniformly chosen from all possible $n!$ permutations, and $1 \le j \le n$. What is $\Pr(|C(1)| = j)$? How many permutations have a cycle of length $n - 2$?

4. There are 100 prisoners numbered $1, \ldots, 100$. The jailor creates 100 opaque boxes numbered from 1 to 100. The jailor also creates 100 cards numbered 1 to 100. Then the cards are put **completely at random** in the boxes, such that each box has exactly one card (hence selecting

a uniform permutation over [100]). Each prisoner is allowed to open 50 boxes sequentially (namely, they can look at what they have seen to decide which box to open next), with the hope of finding their own number. They then put the cards exactly as they found them and close the boxes. If **all of them** find their numbers, they are spared, if not, they are all shot. Is there a strategy such that they all survive with probability at least 0.2? The prisoners **do not** interact with each other during any time.

Problem 3. (Min-cut with node merging). In class we considered min-cut by merging nodes based on randomly picking edges. Suppose instead we pick two random nodes, and merge them at each stage, until we are left with two nodes. Show that there are graphs such that the probability of finding the min-cut using this process is at most c^{-n} for some $c < 1$.