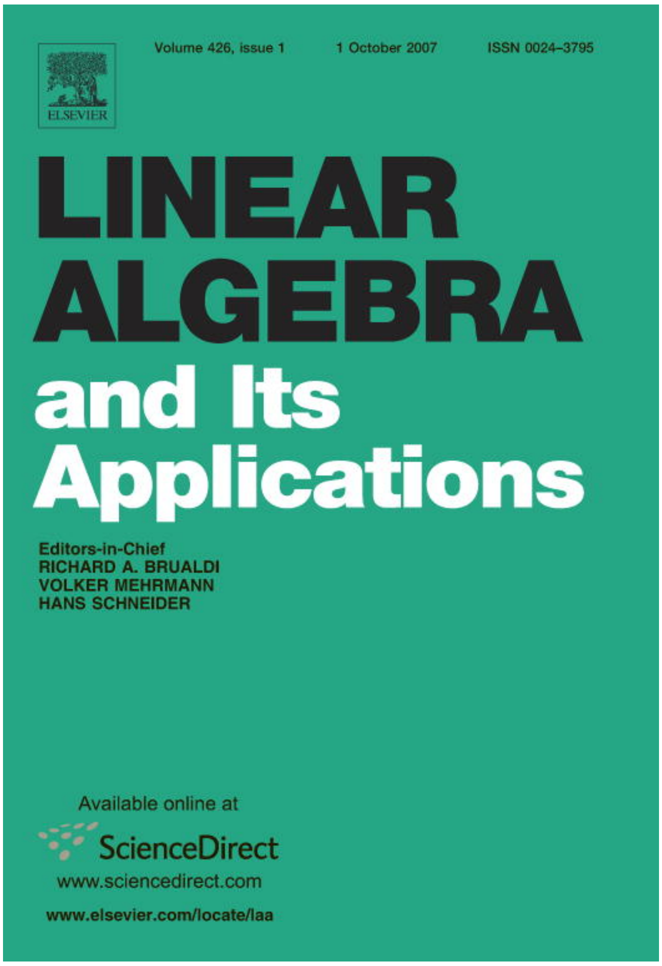
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LINEAR ALGEBRA AND ITS APPLICATIONS

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On the stability of *P*-matrices

A. Kevin Tang [a,*], Alp Simsek [b], Asuman Ozdaglar [b], Daron Acemoglu [b]

[a] *California Institute of Technology, Department of Computer Science, 1200 E. California Blvd., MC 256-80, Pasadena, CA 91125, United States*

[b] *Massachusetts Institute of Technology, United States*

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Abstract

We establish two sufficient conditions for the stability of a *P*-matrix. First, we show that a *P*-matrix is positive stable if its skew-symmetric component is sufficiently smaller (in matrix norm) than its symmetric component. This result generalizes the fact that symmetric *P*-matrices are positive stable, and is analogous to a result by Carlson which shows that sign symmetric *P*-matrices are positive stable. Second, we show that a *P*-matrix is positive stable if it is strictly row (column) square diagonally dominant for every order of minors. This result generalizes the fact that strictly row diagonally dominant *P*-matrices are stable. We compare our sufficient conditions with the sign symmetric condition and demonstrate that these conditions do not imply each other.

Keywords: *P*-matrices; Stability; Symmetry

0. Notations

Given an $n \times n$ matrix A, we denote by A_{ij} its element in ith row and jth column. We denote by $\|A\|_2$ the two-norm of A, that is $\max_{i\in\{1,2,..,n\}} \sqrt{\lambda_i}$ where $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of $A^{\mathrm{T}}A$. For subsets α and β of $\{1, 2.., n\}$, we denote by $A(\alpha|\beta)$ the sub-matrix of A with elements $\{A_{ij}\}$ where $i \in \alpha$ and $j \in \beta$. If $|\alpha| = |\beta|$, then we call $\det(A(\alpha|\beta))$ the minor corresponding to index sets α and β and denote it by $A(\alpha, \beta)$. If $|\alpha| = k$, we call $A(\alpha, \alpha)$ a principal minor of A of order k. Let $\mathbb{R}^N$ denote the N dimensional Euclidean space, and $\mathbb{R}^N_+$ denote the nonnegative

* Corresponding author. Tel.: +1 626 395 3556; fax: +1 626 792 4257.
E-mail address: aotang@caltech.edu (A.K. Tang).

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orthant. Given a complex number $x \in \mathbb{C}$, we denote its polar representation by $x = l(x)\mathrm{e}^{\mathrm{i}\arg(x)}$, where $l(x) \in \mathbb{R}_+$ is the length of x and $\arg(x) \in (-\pi, \pi]$ is its argument.

1. Introduction

A square matrix is called a P-matrix if all of its principal minors are real and positive. Throughout this paper, we focus on real P-matrices. Since their introduction by Fiedler and Ptak [6], P-matrices have found applications in a number of disciplines including physics, economics, and communication networks. To note one example, in economics P-matrices are used to establish sufficient conditions for the uniqueness of the general equilibrium (cf. [1]).[1]

A square matrix is positive stable if all of its eigenvalues lie in the open right half plane. In this paper, we will refer to a positive stable matrix as a stable matrix. Positive stability and related notions of stability are of fundamental importance in linear algebra and they are important in applications too, e.g. in studying dynamical systems in various fields such as control theory, economics, physics, chemical networks, and biology. Gantmacher [7] presents the classic stability results and Hershkowitz [8] provides a recent survey of new results and applications.

P-matrices and stable matrices are closely related. Even though the P-matrix property or stability property do not imply each other for general matrices, they do so for certain well known matrix classes. For example, a symmetric matrix is stable if and only if it is a P-matrix. Moreover, totally nonnegative matrices (i.e. square matrices with every minor positive), nonsingular M-matrices (i.e. matrices with positive diagonal and non-positive off-diagonal entries), and positive definite matrices have both the stability and the P-matrix properties. These observations led to a number of studies which investigated additional conditions that guarantee stability of a P-matrix.

One such study is the earlier work by Carlson [4]. A matrix A is *sign symmetric* if

$$A(\alpha, \beta)A(\beta, \alpha) \geqslant 0 \tag{1}$$

for all $\alpha, \beta \subset \{1, 2, .., n\}$ such that $|\alpha| = |\beta|$. Carlson [4] proved the following:[2]

Theorem 1. *A sign-symmetric P-matrix is stable.*

In this paper, we provide two new sufficient conditions for the stability of a P-matrix along the lines of Theorem 1. Our first result is motivated by the fact that symmetric P-matrices are stable. Since eigenvalues are continuous functions of matrix entries, one may expect that a symmetric P-matrix will remain stable if it is slightly perturbed into a non-symmetric matrix. We formalize this idea by showing that an $n \times n$ P-matrix A is stable if the norm of the matrix $(A^+)^{-1}A^-$ is not greater than $\sin(\pi/n)$, where A^+ represents the symmetric component of A, and A^- the skew-symmetric component. The proof uses a recent result by Eisenstat and Ipsen [5] which bounds the relative change in eigenvalues resulting from a perturbation. We show that the sufficient condition we provide does not imply, nor is it implied by Carlson's sign symmetry condition. Both conditions put symmetry restrictions on a P-matrix to guarantee stability; however, they emphasize different

[1] See also Parthasarthy [12] for other economic applications, Babenko and Petrov [3] for a physics application, Tang et al. [14] for an application in communication networks, and Simsek et al. [13] for an application for mixed and nonlinear complementarity problems.

[2] Carlson [4] also conjectured a stronger theorem. A matrix is weakly sign symmetric if Eq. (1) holds for all $\alpha, \beta \subset \{1, 2, .., n\}$ such that $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$,that is, if the products of symmetrically located almost principal minors are nonnegative. Then, Carlson conjectured that a weakly sign symmetric P-matrix is stable. This conjecture, among some other conjectures, has been disproven by Holtz [10].

aspects of symmetry. Finally, in terms of computational complexity, this sufficient condition is much easier to check than the sign symmetry condition.

Our second result is motivated by the fact that strictly diagonally dominant P-matrices are stable. An $n \times n$ matrix A is *strictly row diagonally dominant* if

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}|, \quad \text{for } i = 1, \ldots, n, \tag{2}$$

and A is *strictly column diagonally dominant* if A^{T} is strictly row diagonally dominant. It follows from the celebrated Gersgorin's theorem [9] that a strictly row or column diagonally dominant P-matrix is stable. We show that a potentially weaker diagonal dominance condition would be sufficient to guarantee the stability of a P-matrix. We say that a matrix A is *strictly row square diagonally dominant for every order of minors* if for any $\alpha \in \{1, 2, .., n\}$,

$$A(\alpha, \alpha)^2 > \sum_{|\beta|=|\alpha|, \beta \neq \alpha, \beta \in \{1,2,\ldots,n\}} A(\alpha, \beta)^2. \tag{3}$$

We show that a P-matrix which also satisfies this condition is stable. Our proof of this theorem is analogous to the proof of Theorem 1 in Carlson [4]. Moreover, this sufficient condition and Carlson's sign symmetry condition both involve computing all minors of a matrix. Nevertheless, this condition too does not imply, nor is it implied by Carlson's sign symmetry condition. Intuitively, this condition guarantees stability by restricting the size of off-diagonal entries of the matrix, which is essentially different from our first condition and Carlson's sign symmetry condition which establish stability through symmetry restrictions.

The organization of this paper is as follows. In Section 2, we state and prove our first theorem which establishes stability when the P-matrix is "almost" symmetric. In Section 3, we state and prove our second theorem which establishes stability for P-matrices that are strictly row (column) square diagonally dominant for every order of minors. In this section, we also provide a weaker diagonal dominance condition which guarantees stability of a P-matrix in lower dimensions. In Section 4, we compare our conditions with Carlson's sign symmetry condition. In particular, we provide examples which show that these conditions do not imply each other. We also present a numerical exercise to calculate how likely each condition is to be encountered if matrices are generated randomly according to certain probability measures.

2. Stability of almost symmetric P-matrices

In this section, we provide a sufficient condition for the stability of P-matrices that are close to being symmetric. Our result in this section holds for the set of Q-matrices, which is a superset of P-matrices. A square matrix is called a Q-matrix if for each $k \in \{1, 2, .., n\}$, its sum of principal minors of order k is positive. Clearly any P-matrix is also a Q-matrix.

We need the following lemma to prove our results in both this and the next section. The result, due to Kellogg [11], states that every eigenvalue of a Q-matrix lies in a particular open angular wedge.

Lemma 1. *Let A be an $n \times n$ Q-matrix, and let μ be one of its eigenvalues. Then,*

$$|\arg(\mu)| < \pi - \pi/n.$$

Our main result in this section makes use of the symmetric and the skew-symmetric components of a matrix. Given matrix A, we define its symmetric component as

$$A^+ = \frac{A + A^{\mathrm{T}}}{2},$$

and its skew-symmetric component as

$$A^- = \frac{A - A^{\mathrm{T}}}{2}.$$

If A^+ is much larger than A^- in an appropriate matrix norm, then A will be close to being symmetric. Since eigenvalues change continuously in matrix entries, one might then expect A to be stable in view of the result that symmetric P-matrices are stable. The following theorem, which is our main result in this section, makes this idea precise.

Theorem 2. *Let A be a Q-matrix, A^+ be its symmetric part and A^- be its skew-symmetric part. If A^+ is nonsingular and*

$$\|(A^+)^{-1}A^-\|_2 \leqslant \sin\left(\frac{\pi}{n}\right), \tag{4}$$

then A is stable.

To prove this theorem, we need to characterize how eigenvalues change when the symmetric matrix A^+ is perturbed. The well known Bauer–Fike Theorem [2] in numerical error analysis provides a bound for the absolute error of the change of eigenvalues in response to a perturbation. Recently, the following lemma by Eistenstat and Ipsen [5] provides bounds for the relative error for change in eigenvalues due to this kind of perturbation.[3]

Lemma 2. *Let B be a non-singular and diagonalizable matrix with an eigendecomposition $B = X\Lambda X^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the eigenvalue matrix and X is an eigenvector matrix. Let E be a perturbation matrix of the same size as B and $\hat{\lambda}$ be an eigenvalue of $B + E$. Then*

$$\min_{i\in\{1,2,..,n\}} \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leqslant \kappa(X)\|B^{-1}E\|_2, \tag{5}$$

where $\kappa(X) = \|X\|_2\|X^{-1}\|_2$ is the condition number of the eigenvector matrix X.

Proof of Theorem 2. Since A^+ is real and symmetric, it is diagonalizable. Furthermore, there exists a unitary eigenvector matrix T, such that

$$A^+ = T\Lambda T^{-1},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the eigenvalue matrix of A^+. Since T is unitary, $\|T\|_2 = 1$ and $\|T^{-1}\|_2 = 1$ which implies $\kappa(T) = 1$. Let μ be an eigenvalue of A. Then, since $A = A^+ + A^-$, by Lemma 2, we have

$$\min_i \frac{|\lambda_i - \mu|}{|\lambda_i|} \leqslant \|(A^+)^{-1}A^-\|_2. \tag{6}$$

Let $\lambda(\mu)$ be a minimizer of the problem (6), $r(\mu)$ be the positive scalar given by

$$r(\mu) = \|(A^+)^{-1}A^-\|_2|\lambda(\mu)|, \tag{7}$$

[3] Results similar to Lemma 1 hold for norms other than the two-norm, e.g. Frobenius norm (see [5]). Consequently, it is possible to prove results analogous to Theorem 2 using other norms. Here we state and prove results for the two-norm case to simplify the exposition.

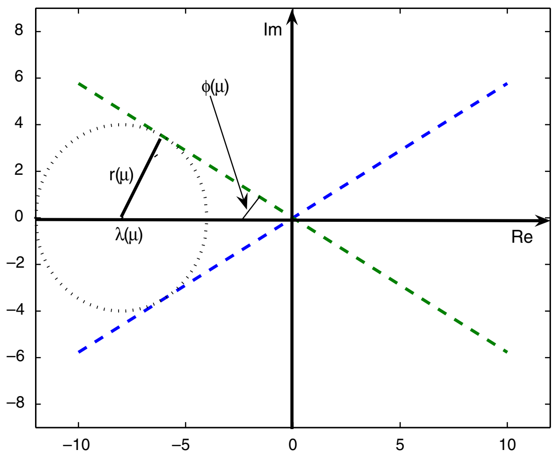

Fig. 1. Locations of eigenvalues.

and $B(\mu)$ be a ball around $\lambda(\mu)$ with radius $r(\mu)$, i.e.

$$B(\mu) = \{x \in \mathbb{C} | |x - \lambda(\mu)| \leqslant r(\mu)\}.$$

Then, Eq. (6) is equivalent to saying $\mu \in B(\mu)$.

For $0 < \phi < \pi/2$, let the set $C(\phi)$ be given by

$$C(\phi) = \{x = r\mathrm{e}^{\mathrm{j}\theta}|_{r\in\mathbb{R}_+,\theta\in(-\pi,\pi]} \in \mathbb{C} | |\theta| < \phi \text{ or } |\theta| > \pi - \phi\}.$$

Note that $C(\phi)$ represents two cones that are symmetric with respect to the imaginary axis. By Eqs. (4) and (7),

$$\sin(\phi(\mu)) = \frac{r(\mu)}{|\lambda(\mu)|} \leqslant \sin\left(\frac{\pi}{n}\right),$$

moreover, since A^+ is symmetric, $\lambda(\mu)$ is real. Then

$$B(\mu) \subset C\left(\frac{\pi}{n}\right),$$

(cf. Fig. 1) which implies

$$\mu \in C\left(\frac{\pi}{n}\right).$$

On the other hand, by Lemma 1, $|\arg(\mu)| < \pi - \pi/n$, hence μ cannot be in the left cone of the set $C(\phi)$. This implies

$$|\arg(\mu)| < \frac{\pi}{n},$$

showing, in particular, that A is stable. □

3. Stability of almost diagonally dominant P-matrices

In this section, we provide a sufficient condition for the stability of P-matrices that are close to being row or column diagonally dominant. To state our result, we introduce a new notion of diagonal dominance. A matrix A is *strictly row square diagonally dominant for every order of minors* if Eq. (3) holds for any $\alpha \subset \{1, 2, .., n\}$. A is *strictly column square diagonally dominant for every order of minors* if A^{T} is strictly row square diagonally dominant for every order of minors. Condition (3) can be potentially weaker than the row diagonal dominance condition given in (2); see, for example, the matrix in example 5. The following result asserts that Condition (3) is enough to guarantee stability of P-matrices.

Theorem 3. *Let A be a P-matrix. If A is strictly row (column) square diagonally dominant for every order of minors, then A is stable.*

Proof. We prove the theorem for the row dominant case. The proof for the column dominant case is analogous. Since A is a P-matrix, there exists a $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i > 0$ for $i = 1, 2, \ldots, n$ such that AD is positive stable (cf. [9]). For $t \in [0, 1]$, let the matrix $D(t)$ be given by

$$D(t) = tD + (1 - t)I \tag{8}$$

for $0 \leqslant t \leqslant 1$ and the matrix $S(t)$ by

$$S(t) = (AD(t))^2. \tag{9}$$

We claim that $S(t)$ is a Q matrix. By the Cauchy–Binet formula, we have, for each $\alpha \subset \{1, 2, .., n\}$,

$$\begin{aligned}
S(t)(\alpha,\alpha) &= \sum_{|\beta|=|\alpha|;\beta\subset\{1,2,..,n\}} AD(t)(\alpha,\beta)AD(t)(\beta,\alpha)\\
&= \sum_{|\beta|=|\alpha|;\beta\subset\{1,2,..,n\}} \left(\prod_{i\in\beta} D(t)_{ii}\right)\left(\prod_{i\in\alpha} D(t)_{ii}\right) A(\alpha,\beta)A(\beta,\alpha)\\
&> \sum_{|\beta|=|\alpha|,\beta\neq\alpha;\beta\subset\{1,2,..,n\}} \left(\prod_{i\in\alpha} D(t)_{ii}\right)^2 A(\alpha,\beta)^2\\
&\quad + \left(\prod_{i\in\beta} D(t)_{ii}\right)\left(\prod_{i\in\alpha} D(t)_{ii}\right) A(\alpha,\beta)A(\beta,\alpha),
\end{aligned} \tag{10}$$

where we used Condition (3) to get the inequality. For $k \in \{1, 2, .., n\}$, adding Eq. (10) over all principal minors of $S(t)$ of order k, we have

$$\begin{aligned}
\sum_{|\alpha|=k} S(\alpha,\alpha) &> \sum_{|\alpha|=k}\left(\sum_{|\beta|=|\alpha|,\beta\neq\alpha}\left(\prod_{i\in\alpha} D(t)_{ii}\right)^2 A(\alpha,\beta)^2\right.\\
&\quad \left. + \left(\prod_{i\in\beta} D(t)_{ii}\right)\left(\prod_{i\in\alpha} D(t)_{ii}\right) A(\alpha,\beta)A(\beta,\alpha)\right)\\
&= \sum_{|\alpha|=|\beta|=k,\alpha\neq\beta}\left(\left(\prod_{i\in\alpha} D(t)_{ii}\right)A(\alpha,\beta) + \left(\prod_{i\in\beta} D(t)_{ii}\right)A(\beta,\alpha)\right)^2\\
&> 0,
\end{aligned}$$

showing that $S(t)$ is a Q-matrix for $t \in [0, 1]$. Then, by Lemma 1, $S(t)$ cannot have non-positive real eigenvalues. Since $S(t) = (AD(t))^2$, eigenvalues of $S(t)$ are the squares of eigenvalues of

$AD(t)$, hence $AD(t)$ cannot have eigenvalues on the imaginary axis for any $t \in [0, 1]$. Since the stable matrix $AD(1) = AD$ have all eigenvalues on the right half space and since eigenvalues change continuously in matrix entries, $AD(t)$ has eigenvalues on the right half space for all $t \in [0, 1]$. We conclude, in particular, that $AD(0) = A$ is stable. $\square$

In the remaining of this section, we provide a stronger result than Theorem 3 for low dimensional matrices. We say that a matrix A is *strictly diagonally dominant of its row entries*

$$|A_{ii}| > |A_{ij}| \quad \text{for each } i = 1, \dots, n \text{ and all } j \neq i. \tag{11}$$

A is *strictly diagonally dominant of its column entries* if A^{T} is strictly diagonally dominant of its row entries. Condition (11) is weaker than both Condition (2) and Condition (3). Hence, the following result generalizes Theorem 3 for low dimensional P-matrices.

Theorem 4. *Let A be an $n \times n$ P-matrix where $n \in \{1, 2, 3\}$. If A is strictly diagonally dominant of its row (or column) entries, then it is stable.*

Proof. The result trivially holds for $n \in \{1, 2\}$. We prove it for $n = 3$. For $i \in \{1, .., n\}$, let

$$E_i = \sum_{|\alpha|=i;\alpha\subset\{1,..,n\}} A(\alpha, \alpha)$$

denote the sum of all principal minors of A of order i. By the Routh stability criterion (cf. [7]), A is stable if and only if

$$E_i > 0 \quad \text{for } i \in \{1, 2, 3\}, \tag{12}$$

and

$$E_1 E_2 > E_3. \tag{13}$$

Eq. (12) holds since A is a P-matrix. We note that Eq. (13) also holds since

$$\begin{aligned}
E_3 = \det(A) &= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31}) \\
&< A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} + A_{11}A_{22}A_{33} - A_{12}A_{21}A_{33} + A_{11}A_{22}A_{33} - A_{13}A_{22}A_{31} \\
&= A_{11}(A_{22}A_{33} - A_{23}A_{32}) + A_{22}(A_{11}A_{33} - A_{13}A_{31}) + A_{33}(A_{11}A_{22} - A_{12}A_{21}) \\
&< (A_{11} + A_{22} + A_{33})((A_{11}A_{22} - A_{12}A_{21}) + (A_{11}A_{33} - A_{13}A_{31}) + (A_{22}A_{33} - A_{23}A_{32})) \\
&= E_1 E_2,
\end{aligned}$$

where the first inequality follows since $A_{ii} = |A_{ii}| > |A_{ij}|$ for $j \neq i$ and the second one follows since $A(\{i, j\}, \{i, j\}) = A_{ii}A_{jj} - A_{ij}A_{ji} > 0$ for $j \neq i$. $\square$

Theorem 4 does not generalize to higher dimensions as the following counterexample demonstrates.

Example 1. Let A be the matrix given by

$$A = \begin{bmatrix} 9.4554 & -8.8510 & 4.5878 & 6.2469 \\ -5.1538 & 8.8516 & 4.6208 & -8.6458 \\ -3.6440 & 5.0156 & 7.6989 & 6.7869 \\ 9.4550 & 5.2547 & 7.6980 & 9.4554 \end{bmatrix}.$$

It can be checked that A is a P-matrix and is diagonally dominant of its row and column entries. However, the four eigenvalues of A are $18.2129 + 3.6058\mathrm{i}$, $18.2129 - 3.6058\mathrm{i}$, $-0.4823 + 6.5399\mathrm{i}$ and $-0.4823 + 6.5399\mathrm{i}$, hence A is not stable.

4. Comparison with existing results

We have provided in Theorems 2 and 3 two new general sufficient conditions for the stability of P-matrices. In this section, we compare our sufficient conditions with that of Carlson's sign symmetry condition.

We first demonstrate that these three sufficient conditions do not imply each other. The following series of examples show that for any pair of conditions there exists a matrix which satisfies the assumptions of one but not the other.

Example 2. The following is a stable P-matrix which satisfies the condition of Theorems 2 and 3 but is not sign symmetric:

$$A = \begin{bmatrix} 1 & -0.5 \\ 0.5 & 1 \end{bmatrix}.$$

A is a stable P-matrix with eigenvalues $1 + 0.5\mathrm{i}$ and $1 - 0.5\mathrm{i}$. A is not sign-symmetric since $A(\{1\}, \{2\})A(\{2\}, \{1\}) = -0.25 < 0$. However, A satisfies the condition of Theorem 2 since $\|(A^+)^{-1}A^-\|_2 = 0.5 < \sin\left(\frac{\pi}{2}\right)$. A also satisfies the condition of Theorem 3.

Example 3. The following is a stable P-matrix which is sign symmetric but does not satisfy the condition of Theorem 2 or Theorem 3:

$$A = \begin{bmatrix} 1 & 9 \\ 0.1 & 1 \end{bmatrix}$$

A is a stable P-matrix with eigenvalues 1.9487 and 0.0513. A is sign-symmetric. However, A doesn't satisfy the condition of Theorem 2 since $\|(A^+)^{-1}A^-\|_2 = 1.2535 > \sin\left(\frac{\pi}{2}\right)$, nor does it satisfy the condition of Theorem 3 since $A(\{1\}, \{2\}) = 9 > A(\{1\}, \{1\}) = A(\{2\}, \{2\})$.

Example 4. The following is a stable P-matrix which satisfies the condition of Theorem 2 but not the condition of Theorem 3:

$$A = \begin{bmatrix} 1 & 0.8 & 0.9 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.9 & 1 \end{bmatrix}.$$

A is a stable P-matrix with eigenvalues 2.5957, 0.1816 and 0.2227. A satisfies the condition of Theorem 2 since $\|(A^+)^{-1}A^-\|_2 = 0.4821 < 0.866 = \sin\left(\frac{\pi}{3}\right)$. However, A does not satisfy the condition of Theorem 3 since $A^2(\{1\}, \{1\}) = 1 < 1.45 = A^2(\{1\}, \{2\}) + A^2(\{1\}, \{3\})$ and $A^2(\{1\}, \{1\}) = 1 < 1.13 = A^2(\{2\}, \{1\}) + A^2(\{3\}, \{1\})$.

Example 5. A stable P-matrix that satisfies the condition of Theorem 3 but not the condition of Theorem 2:

Table 1
Conditions to guarantee different properties for a matrix defined in (14)

Properties	Conditions
A is a P-matrix	$xy < 1$
A is a sign-symmetric matrix	$xy \geqslant 0$
A is a matrix that satisfies Theorem 2's $\\|(A^+)^{-1}A^-\\|_2 < \sin\left(\frac{\pi}{2}\right)$	$\|x+y+2\| \geqslant \|x+y-2\|$ and $\|x-y\| < \|x+y-2\|$ or $\|x+y+2\| \leqslant \|x+y-2\|$ and $\|x-y\| < \|x+y+2\|$
A is a matrix that satisfies Theorem 3's $A^2(\alpha,\alpha) > \sum_{\|\beta\|=\|\alpha\|,\beta\neq\alpha} A^2(\alpha,\beta)$	$\|x\| < 1$ and $\|y\| < 1$

$$A = \begin{bmatrix} 1 & 0.6 & 0.7 \\ -0.6 & 1 & 0.5 \\ -0.5 & -0.4 & 1 \end{bmatrix}.$$

A is a stable P-matrix with eigenvalues 0.5524, $1.2238 + 0.8126\mathrm{i}$ and $1.2238 - 0.8126\mathrm{i}$. It can be checked that A is strictly row square diagonal dominant for every order of minors, hence A satisfies the condition of Theorem 3. However, A does not satisfy the condition of Theorem 2 since $\|(A^+)^{-1}A^-\|_2 = 1.0658 > 0.866 = \sin\left(\frac{\pi}{3}\right)$.

We next highlight the similarities and the differences between the three conditions by stating each condition in terms of matrix entries for a general 2×2 matrix. Let

$$A = \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix}. \tag{14}$$

Table 1 summarizes the conditions for A to satisfy the conditions of Theorem 2, Theorem 3, and the sign symmetry condition.[4] Fig. 2 provides an illustration. In general, a matrix is quite symmetric if its diagonal entries are much larger than the off diagonal ones (hence $\|A^+\|$ is much larger than $\|A^-\|$). Therefore, including the condition of Theorem 3, all three conditions impose certain symmetry conditions. the main point here is to show that different conditions emphasize different kinds of "symmetry". For example, sign symmetry tends to emphasize some quadrants and ignore others while Theorem 3 touches all quadrants but only covers cases when the off diagonal terms are small. These intuitions generalize to higher dimensions.

We finally engage in a numerical exercise to get a sense of how likely each of these conditions are to be encountered in applications. To this end, we generate 3×3 matrices drawing entries according to a probability measure μ and we calculate the probability mass of P-matrices satisfying each of the conditions along with the probability mass of all stable P-matrices. The simulation is done by using Monte Carlo method. For each case, 10^6 stable P-matrices are generated. Let the matrix sets S and $S_i|_{i\in\{1,2,3,4\}}$ be given by

S: the set of 3×3 stable P-matrices.
S_1: the set of matrices in S which satisfy the sign symmetry condition.
S_2: the set of matrices in S which satisfy the condition of Theorem 2.
S_3: the set of matrices in S which satisfy the conditions of Theorem 3.
S_4: the set of matrices in S which satisfy the conditions of Theorem 4.

[4] Note that for a 2 by 2 case, a P-matrix is always stable and hence all three sufficient conditions are not so critical for 2 by 2 cases. However, the main point here is to show that different conditions emphasize different kinds of "symmetry".

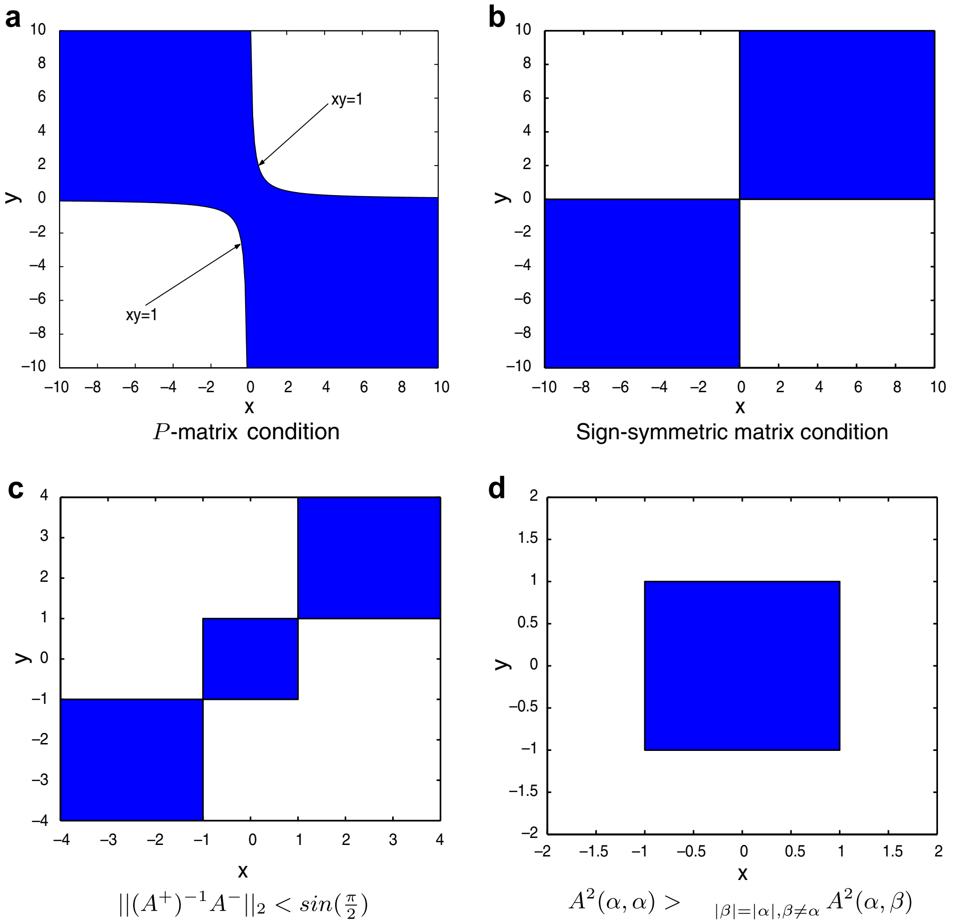

Fig. 2. Graphical illustration of conditions that guarantee certain properties of matrix A [cf. Eq. (14)].

Then $\Pr(S_i|S) = \mu(S_i)/\mu(S)$ represents the probability of the corresponding condition holding given that the matrix is stable. These probabilities are summarized in Table 2. Even though the probabilities depend on the distribution which generates the matrix, it is still clear that the probability of any pair of conditions being satisfied is relatively low, emphasizing the fact that

Table 2
Percentage of 3 by 3 stable P-matrices that are covered by various different conditions

Distributions	Uniform $U(-1,1)$	Gaussian $N(0,1)$	Gaussian $N(1,1)$	Exponential $E(1)$
$\Pr(S_1\|S)$	1.3%	1.7%	6.3%	22.8%
$\Pr(S_2\|S)$	2.1%	3.7%	8.7%	16.0%
$\Pr(S_3\|S)$	3.3%	6.2%	11.3%	21.2%
$\Pr(S_4\|S)$	13.5%	14.0%	28.6%	31.9%
$\Pr(S_1 \cap S_2\|S)$	0.2%	0.3%	1.4%	5.8%
$\Pr(S_1 \cap S_3\|S)$	0.2%	0.3%	1.3%	6.9%
$\Pr(S_2 \cap S_3\|S)$	1.6%	3.2%	6.6%	12.6%

the conditions guarantee stability by focusing on different aspects of matrix entries, namely: sign symmetry, symmetry, and diagonal dominance. Moreover, not surprisingly, $\Pr(S_4|S)$ is larger than $\Pr(S_i|S)$ for $i \in \{1, 2, 3\}$ as Theorem 4 is set up by exploring finer structures and therefore stronger.

5. Conclusions

We have obtained two new general conditions which ensure stability of P-matrices. The first condition is stated in terms of the symmetric part and the skew-symmetric part of a matrix, which is intuitive and easier to check. The second condition asserts that if a P-matrix is strictly row (column) square diagonally dominant for every order of minors, then it is stable. It is further shown that a P-matrix with no more than three dimensions is stable if it is diagonally dominant of its row (column) entries. This implies further stronger theorems may be obtained if we can make good use of stronger tools like the Routh criterion.

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