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DUALITY GAP ESTIMATION VIA A REFINED SHAPLEY–FOLKMAN LEMMA*

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Abstract. Based on concepts like the kth convex hull and finer characterization of nonconvexity of a function, we propose a refinement of the Shapley–Folkman lemma and derive a new estimate for the duality gap of nonconvex optimization problems with separable objective functions. We apply our result to the network utility maximization problem in networking and the dynamic spectrum management problem in communication as examples to demonstrate that the new bound can be qualitatively tighter than the existing ones. The idea is also applicable to cases with general nonconvex constraints.

Key words. nonconvex optimization, duality gap, convex relaxation, network resource allocation

AMS subject classifications. 90C26, 90C46

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1. Introduction.

The Shapley–Folkman lemma (Theorem 1.1) was stated and used to establish the existence of approximate equilibria in economy with nonconvex preferences [13]. It roughly says that the sum of a large number of sets is close to convex and thus can be used to generalize results on convex objects to nonconvex ones.

THEOREM 1.1. *Let $S_1, S_2, \ldots, S_n$ be subsets of $\mathbb{R}^m$. For each $z \in \operatorname{conv} \sum_{i=1}^n S_i = \sum_{i=1}^n \operatorname{conv} S_i$, there exist points $z^i \in \operatorname{conv} S_i$ such that $z = \sum_{i=1}^n z^i$ and $z^i \in S_i$ except for at most m values of i.*

Remark 1.2. In this paper, we use superscripts to index vectors and subscripts to refer to a particular component of a vector. For instance, both x^i and x^{ij} are vectors, but x^i_s is the sth component of the vector x^i. For two vectors x and y, $x \leq y$ means $x_s \leq y_s$ holds for all components.

The Shapley–Folkman lemma has found applications in many fields, including economics and optimization theory. It is of particular use for estimating the duality gap of a general nonconvex optimization problem, which provides an indication of the nonconvexity of such a problem [8, 5, 4]. Such an estimation has attracted much interest in previous research, mainly because of its relation to the approximation algorithms based on dual methods. In this paper, we consider nonconvex optimization problems with separable objectives and linear constraints:

$$\begin{aligned} \min \quad & \sum_{i=1}^n f_i(x^i) \\ \text{s.t.} \quad & \sum_{i=1}^n A_i x^i \leq b. \end{aligned} \tag{1.1}$$

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Here $x^i \in \mathbb{R}^{n_i}$ are the decision variables. The function $f_i : \mathbb{R}^{n_i} \to \bar{\mathbb{R}}$ is proper[1] and lower semicontinuous, and its domain is bounded. A_i is a matrix of size $m \times n_i$, so there are m linear constraints in total. The Lagrange dual problem of (1.1) is

$$\begin{aligned} \max \quad & -\sum_{i=1}^{n} f_i^*(-A_i^T y) - b^T y \\ \text{s.t.} \quad & y \geq 0, \end{aligned} \tag{1.2}$$

where f_i^* is the conjugate function of f_i. In this paper, we always assume the feasibility on the primal problem (1.1). Furthermore, under our assumptions on the functions f_i, the dual problem (1.2) is guaranteed to be feasible, and we denote the optimal value of the primal problem (1.1) and dual problem (1.2) as p and d, respectively. In general, there will be a positive duality gap $p - d > 0$ if some function f_i is not convex.

The optimization (1.1) provides the framework for many important problems in fields such as communication [12] and machine learning [16, 1]. The authors of [3] presented the following upper bound for the duality gap of (1.1) based on the Shapley–Folkman lemma:

$$p - d \leq \min\{m+1, n\} \max_{i=1,\ldots,n} \rho(f_i). \tag{1.3}$$

Here $\rho(f)$ is the nonconvexity of a proper function f defined by

$$\rho(f) = \sup \left\{ f\left(\sum_j \alpha_j x^j \right) - \sum_j \alpha_j f(x^j) \right\} \tag{1.4}$$

over all finite convex combinations of points $x^j \in \operatorname{dom} f$, i.e., $f(x^j) < +\infty$, $\alpha_j \geq 0$ with $\sum_j \alpha_j = 1$.

In [14], an improved bound for the duality gap[2] was given by

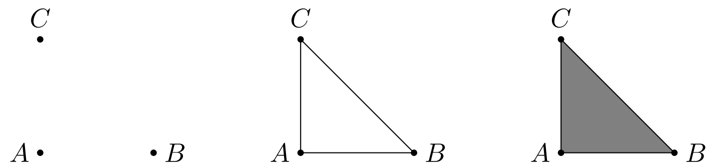
$$p - d \leq \sum_{i=1}^{\min\{m,n\}} \rho(f_i), \tag{1.5}$$

where we assume that $\rho(f_1) \geq \cdots \geq \rho(f_n)$. Although the bound (1.5) is only a slight improvement over the original bound (1.3) by a factor of $m/(m+1)$, it nevertheless shows that (1.3) can never be tight except for some trivial situations. But as will be demonstrated by the examples in this paper, the bound (1.5) can still be very conservative.

In this paper, we aim at providing a tighter duality gap estimation via refining the original Shapley–Folkman lemma. The refined Shapley–Folkman lemma is stated and proved in section 2. Unlike (1.3) and (1.5), our new bound for the duality gap depends on some finer characterization of the nonconvexity of a function, which is introduced in section 3. The new bound itself is given in section 4, which can easily

[1] Here $\bar{\mathbb{R}} = [-\infty, +\infty]$. A function is said to be proper if it never takes $-\infty$ and its domain is nonempty.

[2] In fact, the bound (1.5) derived in [14] is for the difference $p - \hat{p}$, in which $\hat{p}$ is the optimal value of the convexified problem where the function f_i is substituted by f_i^{**}. However, $\hat{p} = d$ because the refined Slater's condition holds for the convexified problem.



(a) $\operatorname{conv}_1 S$ (b) $\operatorname{conv}_2 S$ (c) $\operatorname{conv}_3 S$

FIG. 1. *The kth convex hull of a three-point set $S = \{A, B, C\}$.*

recover existing ones like (1.5). In section 5, we apply it to two examples, a network utility maximization problem in networking and the dynamic spectrum management problem in communication, to demonstrate that the new bound can be qualitatively tighter than the bound (1.5).

If the domain of some function f_i in (1.1) is not convex, by definition $\rho(f_i) = +\infty$. In this case, all the above bounds and our new bound in section 4 will be useless. To handle this issue, we can replace the nonconvexity of the domain by appropriate nonconvex constraints. Although we mainly focus on the case of linear constraints, section 6 shows how the major idea in this paper can be applied to the cases with general convex or even nonconvex constraints.

2. Refined Shapley–Folkman lemma.

To write down our refined version of the Shapley–Folkman lemma, we need to first introduce the concept of the kth convex hull.

DEFINITION 2.1. *The* kth convex hull *of a set S, denoted by* $\operatorname{conv}_k S$, *is the set of convex combinations of k points in S, i.e.,*

$$\operatorname{conv}_k S = \left\{ \sum_{j=1}^{k} \alpha_j v^j \,\middle|\, v^j \in S, \alpha_j \geq 0 \;\; \forall j = 1, \ldots, k, \sum_{j=1}^{k} \alpha_j = 1 \right\}.$$

Figure 1 gives a simple example to illustrate the definition of the kth convex hull. In Figure 1, the set $S = \{A, B, C\}$, $\operatorname{conv}_1 S = S$, $\operatorname{conv}_2 S$ are the segments AB, BC, and CA, while $\operatorname{conv}_3 S$ is the full triangle which is also the convex hull of set S. In general, Carathéodory's theorem implies that $\operatorname{conv}_{m+1} S = \operatorname{conv} S$ for any set $S \subseteq \mathbb{R}^m$. However, for a particular set, the minimum k such that $\operatorname{conv}_k S = \operatorname{conv} S$ can be smaller than $m + 1$, and this number intuitively reflects how the set is closer to being convex. For instance, if we start from $T = \operatorname{conv}_2 S$, the set in Figure 1(b), then $\operatorname{conv}_k T = \operatorname{conv} T$ for $k = 2$.

Next, we recall the concept of k-extreme points of a convex set, which is a generalization of extreme points.

DEFINITION 2.2. *A point z in a convex set S is called a* k-extreme *point of S if we cannot find $(k + 1)$ independent vectors $d^1, d^2, \ldots, d^{k+1}$ such that $z \pm d^i \in S$.*

According to our definition, if a point is k-extreme, then it is also k'-extreme for $k' \geq k$. For a convex set in $\mathbb{R}^m$, a point is an extreme point if and only if it is 0-extreme, a point is on the boundary if and only if it is $(m - 1)$-extreme, and every point is m-extreme. For example, in Figure 1(c), the vertices A, B, C are 0-extreme, the points on segments AB, BC, and CA are 1-extreme, and all the points are 2-extreme.

Now we can state our refined Shapley–Folkman lemma.

THEOREM 2.3. *Let $S_1, S_2, \ldots, S_n$ be subsets of $\mathbb{R}^m$. Assume z is a k-extreme point of* $\operatorname{conv}\sum_{i=1}^n S_i$*; then there exist integers $1 \le k_i \le k+1$ with $\sum_{i=1}^n k_i \le k+n$ and points $z^i \in \operatorname{conv}_{k_i} S_i$ such that $z = \sum_{i=1}^n z^i$.*

The original Shapley–Folkman lemma (Theorem 1.1) now becomes a direct corollary of Theorem 2.3, since any point $z \in \operatorname{conv}\sum_{i=1}^n S_i$ is an m-extreme point. Applying Theorem 2.3 on this point gives a decomposition $z = \sum_{i=1}^n z^i$ with $z^i \in \operatorname{conv}_{k_i} S_i \subseteq \operatorname{conv} S_i$ and $\sum_{i=1}^n k_i \le m+n$. Then the conclusion in Theorem 1.1 follows because $z^i \in S_i$ if $k_i = 1$, while the number of indices i with $k_i \ge 2$ is bounded by m.

Remark 2.4. Our Theorem 2.3 is similar to the refined version of the Shapley–Folkman lemma proposed in [11]. However, the result in [11] does not take the extremeness of the point into account, which can be regarded as a special case of Theorem 2.3 for $k = m$.

To prove Theorem 2.3, we need the following property of k-extreme points in a polyhedron.

LEMMA 2.5. *Let $P \subseteq \mathbb{R}^m$ be a polyhedron, and let z be a k-extreme point of P. Then there exists a vector $a \in \mathbb{R}^m$ such that the set $\{y \in P | a^T y \le a^T z\}$ is in a k-dimensional affine subspace.*

Proof. Assume that the polyhedron P is represented by $Ax \ge b$. Let $A_=$ be the submatrix of A containing the rows of active constraints for the point z, and let $b_=$ be the vector containing the corresponding constants in b. The dimension of the kernel of $A_=$ is at most k. Otherwise, we can find independent and sufficiently small vectors $d^1, \ldots, d^{k+1}$ such that $A_= d^i = 0$ and $A(z \pm d^i) \ge b$ for $i = 1, \ldots, k+1$. This implies $z \pm d^i \in P$, which contradicts the k-extremeness of point z.

Let a be the vector such that a^T is the sum of all rows in $A_=$. Consider a point y satisfying $Ay \ge b$ and $a^T y \le a^T z$. Since adding all inequalities together in $A_= y \ge b_= = A_= z$ gives $a^T y \ge a^T z$, we must have $A_= y = b_=$. Therefore, y is in the affine subspace defined by $A_= x = b_=$ whose dimension is at most k. ∎

Remark 2.6. In the literature, the point satisfying the conclusion of Lemma 2.5 is called a *k-exposed point*. For a general convex set S, a k-extreme point may fail to be a k-exposed point, although it must be in the closure of the set of k-exposed points if S is compact [2]. For the special case of polyhedra, these two concepts are equivalent, and Lemma 2.5 is a generalization of the well-known result that an extreme point of a polyhedron is the unique minimizer of some linear function.

Proof of Theorem 2.3. Since z is in the convex hull of $\sum_{i=1}^n S_i$, there exists some integer l such that z can be written as

$$z = \sum_{j=1}^{l} \alpha_j \sum_{i=1}^{n} v^{ij}, \tag{2.1}$$

in which $v^{ij} \in S_i$, $\alpha_j \ge 0$, $j = 1, \ldots, l$, and $\sum_{j=1}^l \alpha_j = 1$.

Define $S_i' = \{v^{i1}, \ldots, v^{il}\} \subseteq S_i$; then (2.1) actually tells us that $z \in \operatorname{conv}\sum_{i=1}^n S_i'$, so z must be k-extreme in this polytope that lies in $\operatorname{conv}\sum_{i=1}^n S_i$. By Lemma 2.5, there exists a vector $a \in \mathbb{R}^m$ such that the set

$$\left\{ y \in \operatorname{conv}\sum_{i=1}^n S_i' \,\middle|\, a^T y \le a^T z \right\}$$

is in a k-dimensional affine subspace L of $\mathbb{R}^m$. Without loss of generality, we assume that the subspace

$$L = \{y \in \mathbb{R}^m | y_{k+1} = y_{k+2} = \cdots = y_m = 0\}.$$

Next, consider the following linear program in which β_{ij} are the decision variables:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{l} \beta_{ij} a^T v^{ij} \\ \text{s. t.} \quad & \sum_{i=1}^{n}\sum_{j=1}^{l} \beta_{ij} v_s^{ij} = z_s \quad \forall s = 1, \dots, k, \\ & \sum_{j=1}^{l} \beta_{ij} = 1 \quad \forall i = 1, \dots, n, \\ & \beta_{ij} \geq 0 \quad \forall i = 1, \dots, n,\ \forall j = 1, \dots, l. \end{aligned}$$

Setting $\beta_{ij} = \alpha_j$ gives a feasible solution to the above problem with objective value $a^T z$. Among all the optimal solutions, pick up a particular vertex solution β^*_{ij}, which should have at least nl active constraints. We already have $k + n$ active constraints, so the number of nonzero β^*_{ij} entries is at most $k + n$. Define

$$z^i = \sum_{j=1}^{l} \beta^*_{ij} v^{ij}, \quad z' = \sum_{i=1}^{n} z^i,$$

and let k_i be the number of nonzero entries in $\beta^*_{i1}, \dots, \beta^*_{il}$. Since $\sum_{j=1}^{l} \beta^*_{ij} = 1$, there must be a nonzero one and thus $k_i \geq 1$. Now we know that $z^i \in \mathrm{conv}_{k_i} S_i$, and $\sum_{i=1}^{n} k_i \leq k + n$ implies that each k_i cannot exceed $k + 1$. The remaining thing to show is $z_s = z'_s$ for $s = k + 1, \dots, m$. Because

$$z' \in \sum_{i=1}^{n} \mathrm{conv}\, S'_i = \mathrm{conv} \sum_{i=1}^{n} S'_i$$

and

$$a^T z' = \sum_{i=1}^{n}\sum_{j=1}^{l} \beta^*_{ij} a^T v^{ij} \leq a^T z,$$

$z' \in L$. Since $z \in L$, the last $m - k$ components of both z and z' are all zeros, so $z = z'$. ∎

In section 4, when proving the bound for the duality gap, we will not directly apply Theorem 2.3 but a special case of it given by the following Corollary 2.7. At that time, we will see how Corollary 2.7 will improve the bound compared with the existing result such as [11] without the consideration of extremeness.

Corollary 2.7. *Let $S_1, S_2, \dots, S_n$ be subsets of $\mathbb{R}^m$. If $z \in \mathrm{conv} \sum_{i=1}^{n} S_i$, then there exist integers $1 \leq k_i \leq m$ with $\sum_{i=1}^{n} k_i \leq m - 1 + n$ and points $z^i \in \mathrm{conv}_{k_i} S_i$ such that $z_s = \sum_{i=1}^{n} z^i_s$ for $s = 1, \dots, m - 1$ and $z_m \geq \sum_{i=1}^{n} z^i_m$.*

Proof. Using the same argument in the proof of Theorem 2.3, choose $S_i' \subseteq S_i$ containing finite points such that $z \in \operatorname{conv} \sum_{i=1}^n S_i'$. Since $\operatorname{conv} \sum_{i=1}^n S_i'$ is a compact set,

$$\inf\left\{ w_m \,\middle|\, w \in \operatorname{conv} \sum_{i=1}^n S_i', w_1 = z_1, \dots, w_{m-1} = z_{m-1} \right\}$$

can be achieved by some point w^*. w^* is an $(m-1)$-extreme point of $\operatorname{conv} \sum_{i=1}^n S_i'$, and applying Theorem 2.3 on the point w^* gives the desired result. ∎

3. Characterization of nonconvexity. To improve the bound (1.5), some finer characterization of the nonconvexity of a function has to be introduced. In parallel with the definition of the kth convex hull of a set, define the *kth nonconvexity* $\rho^k(f)$ of a proper function f to be the supremum in (1.4) taken over the convex combinations of k points instead of an arbitrary number of points. Obviously,

$$0 = \rho^1(f) \le \rho^2(f) \le \cdots \le \rho(f).$$

For functions f with nonconvex domain, $\rho^k(f) = +\infty$ for $k > 1$. In general, we have the following property.

PROPOSITION 3.1. *For any proper function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, $\rho^{n+1}(f) = \rho(f)$.*

Proof. We only need to show that $\rho(f) \le \rho^{n+1}(f)$. Choose any convex combination $x = \sum_{j=1}^l \alpha_j x^j$ with all points $x^j \in \operatorname{dom} f$, $\alpha_j \ge 0$, and $\sum_{j=1}^l \alpha_j = 1$. Since $(x^j, f(x^j)) \in \operatorname{epi} f$, the point

$$\left(\sum_{j=1}^l \alpha_j x^j, \sum_{j=1}^l \alpha_j f(x^j) \right) \in \operatorname{conv} \operatorname{epi} f.$$

Using Corollary 2.7 on a single set $S_1 = \operatorname{epi} f$, we can find $(y^i, t_i) \in \operatorname{epi} f$, $\beta_i \ge 0$, $i = 1, \dots, n+1$, and $\sum_{i=1}^{n+1} \beta_i = 1$ such that

$$x = \sum_{j=1}^l \alpha_j x^j = \sum_{i=1}^{n+1} \beta_i y^i, \quad \sum_{j=1}^l \alpha_j f(x^j) \ge \sum_{i=1}^{n+1} \beta_i t_i.$$

Now

$$f(x) - \sum_{j=1}^l \alpha_j f(x^j) \le f\left(\sum_{i=1}^{n+1} \beta_i y^i \right) - \sum_{i=1}^{n+1} \beta_i t_i \le f\left(\sum_{i=1}^{n+1} \beta_i y^i \right) - \sum_{i=1}^{n+1} \beta_i f(y^i),$$

which implies $\rho(f) \le \rho^{n+1}(f)$. ∎

For lower semicontinuous functions, the following proposition provides an equivalent definition for the kth nonconvexity, which sheds light on the connection between the concepts of the kth nonconvexity and kth convex hull.

PROPOSITION 3.2. *Assume a proper function f is lower semicontinuous and bounded below by some affine function. Let $f^{(k)}$ be the function whose epigraph is the closure of the kth convex hull of the epigraph of f, i.e.,*

$$\operatorname{epi} f^{(k)} = \operatorname{cl} \operatorname{conv}_k \operatorname{epi} f.$$

Then

$$\rho^k(f) = \sup_x \{f(x) - f^{(k)}(x)\}, \tag{3.1}$$

where we interpret $(+\infty) - (+\infty) = 0$.

Proof. The assumption on the function f implies that $f^{(k)}$ is also a proper function. Consider an arbitrary k-point convex combination of points $x^j \in \operatorname{dom} f$ for $j = 1, \ldots, k$. Following the first step in the proof of Proposition 3.1, we have

$$\left(\sum_{j=1}^k \alpha_j x^j, \sum_{j=1}^k \alpha_j f(x^j) \right) \in \operatorname{conv}_k \operatorname{epi} f \subseteq \operatorname{cl} \operatorname{conv}_k \operatorname{epi} f = \operatorname{epi} f^{(k)}.$$

Therefore,

$$f\left(\sum_{j=1}^k \alpha_j x^j \right) - \sum_{j=1}^k \alpha_j f(x^j) \le f\left(\sum_{j=1}^k \alpha_j x^j \right) - f^{(k)}\left(\sum_{j=1}^k \alpha_j x^j \right),$$

which implies

$$\rho^k(f) \le \sup_x \{f(x) - f^{(k)}(x)\}.$$

To prove the reverse direction, for any $x \in \operatorname{dom} f^{(k)}$,

$$(x, f^{(k)}(x)) \in \operatorname{epi} f^{(k)} = \operatorname{cl} \operatorname{conv}_k \operatorname{epi} f.$$

In the case of $x \in \operatorname{dom} f$, by the lower semicontinuity of f, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(y) \ge f(x) - \epsilon \quad \forall \|y - x\| < \delta.$$

There exists $(\kappa, \eta) \in \operatorname{conv}_k \operatorname{epi} f$ which is sufficiently close to $(x, f^{(k)}(x))$ such that

$$\|\kappa - x\| < \delta, \quad \eta \le f^{(k)}(x) + \epsilon.$$

Because $(\kappa, \eta) \in \operatorname{conv}_k \operatorname{epi} f$, there exists $\alpha_j \ge 0$ for $j = 1, \ldots, k$ such that $\sum_{j=1}^k \alpha_j = 1$ and

$$\kappa = \sum_{j=1}^k \alpha_j x^j, \quad \eta \ge \sum_{j=1}^k \alpha_j f(x^j)$$

in which $x^j \in \operatorname{dom} f$. Thus

$$f(x) - f^{(k)}(x) \le f(\kappa) - \eta + 2\epsilon \le f\left(\sum_{j=1}^k \alpha_j x^j \right) - \sum_{j=1}^k \alpha_j f(x^j) + 2\epsilon \le \rho^k(f) + 2\epsilon.$$

Consider the other case of $x \notin \operatorname{dom} f$, i.e., $f(x) = +\infty$. By the lower semicontinuity of f, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(y) \ge 1/\epsilon \quad \forall \|y - x\| < \delta.$$

We choose $(\kappa, \eta) \in \operatorname{conv}_k \operatorname{epi} f$ as in the above case. Then

$$1/\epsilon - f^{(k)}(x) \leq f(\kappa) - \eta + \epsilon \leq f\left(\sum_{j=1}^{k} \alpha_j x^j\right) - \sum_{j=1}^{k} \alpha_j f(x^j) + \epsilon \leq \rho^k(f) + \epsilon.$$

In both cases, we get $f(x) - f^{(k)}(x) \leq \rho^k(f)$ by letting $\epsilon \to 0$. ∎

Remark 3.3. If a proper function f is bounded below by some affine function, then

$$\operatorname{epi} f^{**} = \operatorname{cl} \operatorname{conv} f$$

(see [10, Theorem X.1.3.5]). Therefore, (3.1) can be regarded as a generalization for the alternative definition of nonconvexity

$$\rho(f) = \sup_x \{f(x) - f^{**}(x)\}$$

used in [14].

In the remainder of this section, three examples will be given to illustrate how to calculate the kth nonconvexity of a particular function. The results in Examples 3.4 and 3.5 will be used by the network utility maximization problem in section 5.1, and Example 3.6 will be used by the dynamic spectrum management problem in section 5.2.

Example 3.4. Consider the function

$$f(x) = f(x_1, \ldots, x_n) = \min_{s=1,\ldots,n} x_s$$

defined on the box $0 \leq x \leq 1$, $x \in \mathbb{R}^n$. It is already known that $\rho(f) = (n-1)/n$ (see [14, Table 1]). By Proposition 3.1, $\rho^k(f) = \rho(f) = (n-1)/n$ for $k \geq n+1$.

For $k = 1, \ldots, n$, as in the proof of Proposition 3.2, pick up any k-point convex combination of points $0 \leq x^j \leq 1$, $j = 1, \ldots, k$. For a given $i \in \{1, \ldots, k\}$, let $s(i)$ be the index such that $x^i_{s(i)}$ is the minimum among $x^i_1, \ldots, x^i_n$. Then

$$\begin{aligned} f(x) = \min_{s=1,\ldots,n} \left\{\sum_{j=1}^{k} \alpha_j x^j_s\right\} &\leq \sum_{j=1}^{k} \alpha_j x^j_{s(i)} \\ &\leq \alpha_i x^i_{s(i)} + 1 - \alpha_i = \alpha_i f(x^i) + 1 - \alpha_i, \end{aligned}$$

where we use the fact that all x^j are within the box $0 \leq x \leq 1$. Summing up among $i = 1, \ldots, k$, we have

$$kf(x) \leq \sum_{i=1}^{k} \alpha_i f(x^i) + k - 1,$$

which implies

$$f(x) - \sum_{i=1}^{k} \alpha_i f(x^i) \leq \frac{k-1}{k}\left(1 - \sum_{i=1}^{k} \alpha_i f(x^i)\right) \leq \frac{k-1}{k}.$$

The above argument shows that $\rho^k(f) \le (k-1)/k$. In fact, the equality holds, which can be easily seen by considering the average of first k points of

$$\begin{aligned} x^1 &= (0, 1, \dots, 1), \\ x^2 &= (1, 0, \dots, 1), \\ &\dots, \\ x^n &= (1, 1, \dots, 0). \end{aligned}$$

In conclusion,

$$\rho^k(f) = \begin{cases} \dfrac{k-1}{k} & \text{if } k = 1, \dots, n, \\ \dfrac{n-1}{n} & \text{if } k \ge n+1. \end{cases}$$

Example 3.5. Consider the function

$$g(x) = g(x_1, \dots, x_n) = -\log \max_{s=1,\dots,n} x_s$$

defined on the region $x \ge 0$ except $x = 0$.

For $k = 1, \dots, n$, pick up any k-point convex combination. Without loss of generality, assume the coefficients $\alpha_j > 0$ for $j = 1, \dots, k$. For a given $i \in \{1, \dots, k\}$, let $s(i)$ be the index such that $x^i_{s(i)}$ is the maximum among $x^i_1, \dots, x^i_n$. Then

$$\begin{aligned} g(x) &= -\log \max_{s=1,\dots,n} \left\{ \sum_{j=1}^k \alpha_j x^j_s \right\} \le -\log \sum_{j=1}^k \alpha_j x^j_{s(i)} \\ &\le -\log(\alpha_i x^i_{s(i)}) = -\log \alpha_i + g(x^i). \end{aligned}$$

Summing up among $i = 1, \dots, k$ with weight α_i, we have

$$\begin{aligned} g(x) &\le -\sum_{i=1}^k \alpha_i \log \alpha_i + \sum_{i=1}^k \alpha_i g(x^i) \\ &\le \log k + \sum_{i=1}^k \alpha_i g(x^i). \end{aligned}$$

The above argument shows that $\rho^k(g) \le \log k$. In fact, the equality holds, which can be easily seen by considering the average of first k points of

$$\begin{aligned} x^1 &= (1, 0, \dots, 0), \\ x^2 &= (0, 1, \dots, 0), \\ &\dots, \\ x^n &= (0, 0, \dots, 1). \end{aligned}$$

To calculate $\rho^{n+1}(g)$, define $h(x) = -\log \sum_{s=1}^n x_s$. Then $h(x)$ is convex and

$g(x) - \log n \le h(x) \le g(x)$. Thus, for any $(n+1)$-point convex combination,

$$\begin{aligned} g\left(\sum_{j=1}^{n+1} \alpha_j x^j\right) &\le h\left(\sum_{j=1}^{n+1} \alpha_j x^j\right) + \log n \\ &\le \sum_{j=1}^{n+1} \alpha_j h(x^j) + \log n \\ &\le \sum_{j=1}^{n+1} \alpha_j g(x^j) + \log n. \end{aligned}$$

Therefore, $\rho^{n+1}(g) \le \log n$. On the other hand, $\rho^{n+1}(g) \ge \rho^n(g) = \log n$.

In conclusion,

$$\rho^k(g) = \begin{cases} \log k & \text{if } k = 1, \dots, n, \\ \log n & \text{if } k \ge n+1. \end{cases}$$

Example 3.6. Consider the function

$$h_\sigma(x) = h_\sigma(x_1, \dots, x_n) = \sum_{s=1}^{n} \log \frac{\|x\|_1 - x_s + \sigma}{\|x\|_1 + \sigma}$$

defined on the box $0 \le x \le 1$, $x \in \mathbb{R}^n$. Here σ is a parameter in the range $0 < \sigma \le 1$.

For complicated functions such as this one, it is usually hard to compute their kth nonconvexity exactly. However, sometimes we can approximate the kth nonconvexity of a function by reducing it to another function whose nonconvexity is already known. Using this technique, we are able to show that

$$\rho^k(h_\sigma) \le \log(k/\sigma).$$

The details are given in Appendix A.

4. Bounding duality gap. Now we can state the main result on the duality gap between the primal problem (1.1) and the dual problem (1.2).

THEOREM 4.1. *Assume that the primal problem* (1.1) *is feasible, i.e., $p < +\infty$. Then there exist integers $1 \le k_i \le m+1$ such that $\sum_{i=1}^n k_i \le m+n$ and the duality gap*

$$p - d \le \sum_{i=1}^{n} \rho_i^{k_i}.$$

Here $\rho_i^k = \rho^k(f_i)$ is the kth nonconvexity of function f_i.

First, let us define the perturbation function $v : \mathbb{R}^m \to \bar{\mathbb{R}}$ by letting $v(z)$ be the optimal value of the perturbed problem

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} f_i(x^i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} A_i x^i \le b + z. \end{aligned}$$

As is the case with convex optimization, $p = v(0)$ and $d = v^{**}(0)$ (see [9, p. 50, Lemmas 2.2–2.3]).

LEMMA 4.2. *The perturbation function v is lower semicontinuous.*

Proof. Pick any $z \in \mathbb{R}^m$. We want to show that if $z^k \to z$ as $k \to \infty$,

$$l = \liminf_{k\to\infty} v(z^k) \geq v(z).$$

The above inequality clearly holds when $l = +\infty$. If $l < +\infty$, by considering a subsequence of $\{v(z^k)\}_{k=1}^{\infty}$, without loss of generality we can assume $v(z^k) < +\infty$ for each k and

$$\lim_{k\to\infty} v(z^k) = l.$$

For each k, find $(\hat{x}^{1k}, \ldots, \hat{x}^{nk})$ attaining the optimal value of the perturbed problem related to $v(z^k)$, i.e.,

$$v(z^k) = \sum_{i=1}^{n} f_i(\hat{x}^{ik}), \quad \sum_{i=1}^{n} A_i \hat{x}^{ik} \leq b + z^k.$$

By extracting a convergent subsequence for each $\{\hat{x}^{ik}\}_{k=1}^{\infty}$, we can assume $\{\hat{x}^{ik}\}_{k=1}^{\infty}$ has a limit x^i. Then

$$\sum_{i=1}^{n} A_i x^i \leq b + z,$$

which implies that $(x^1, \ldots, x^n)$ is feasible to the perturbed problem related to $v(z)$, so

$$\sum_{i=1}^{n} f_i(x^i) \geq v(z).$$

Now

$$l = \lim_{k\to\infty} \sum_{i=1}^{n} f_i(\hat{x}^{ik}) \geq \sum_{i=1}^{n} \liminf_{k\to\infty} f_i(\hat{x}^{ik}) \geq \sum_{i=1}^{n} f_i(x^i) \geq v(z),$$

because f_i is lower semicontinuous. ∎

Proof of Theorem 4.1. Since (1.1) is feasible, $v(0) = p < +\infty$. Let

$$\xi = \sum_{i=1}^{n} \inf_{x^i} f_i(x^i).$$

Then by our assumption of f_i, ξ is finite. $v(z) \geq \xi$ for all $z \in \mathbb{R}^m$. As a consequence, $v(z)$ is bounded below by some affine function, so

$$-\infty < v^{**}(0) \leq v(0) < +\infty, \quad \operatorname{epi} v^{**} = \operatorname{cl}\operatorname{conv}\operatorname{epi} v.$$

By Lemma 4.2, v is lower semicontinuous. Since

$$(0, v^{**}(0)) \in \operatorname{epi} v^{**} = \operatorname{cl}\operatorname{conv}\operatorname{epi} v,$$

for every $\epsilon > 0$, there exists $(\kappa, \eta) \in \operatorname{conv} \operatorname{epi} v$ which is sufficiently close to $(0, v^{**}(0))$ such that

$$v(\kappa) \geq v(0) - \epsilon, \quad \eta \leq v^{**}(0) + \epsilon. \tag{4.1}$$

Because $(\kappa, \eta) \in \operatorname{conv} \operatorname{epi} v$, there exists some integer l and $\alpha_j \geq 0$ for $j = 1, \dots, l$ such that $\sum_{j=1}^{l} \alpha_j = 1$ and

$$\kappa = \sum_{j=1}^{l} \alpha_j z^j, \quad \eta \geq \sum_{j=1}^{l} \alpha_j v(z^j)$$

in which $z^j \in \operatorname{dom} v$.

For each $j = 1, \dots, l$, find $(\hat{x}^{1j}, \dots, \hat{x}^{nj})$ attaining the optimal value of the perturbed problem related to $v(z^j)$, i.e.,

$$v(z^j) = \sum_{i=1}^{n} f_i(\hat{x}^{ij}), \quad \sum_{i=1}^{n} A_i \hat{x}^{ij} \leq b + z^j,$$

which means there exists some vector $w^j \in \mathbb{R}^m_+$ such that

$$(b + z^j - w^j, v(z^j)) \in \sum_{i=1}^{n} C_i,$$

where

$$C_i = \{(A_i x^i, f_i(x^i)) | f_i(x^i) < +\infty, x^i \in \mathbb{R}^{n_i}\}.$$

Taking convex combination of the points above, we have

$$\left(b + \kappa - \sum_{j=1}^{l} \alpha_j w^j, \sum_{j=1}^{l} \alpha_j v(z^j) \right) \in \operatorname{conv} \sum_{i=1}^{n} C_i.$$

Now we can apply Corollary 2.7,[3] which gives points $(r^i, s_i) \in \operatorname{conv}_{k_i} C_i$ with $1 \leq k_i \leq m + 1$ such that

$$b + \kappa \geq b + \kappa - \sum_{j=1}^{l} \alpha_j w^j = \sum_{i=1}^{n} r^i, \quad \eta \geq \sum_{j=1}^{l} \alpha_j v(z^j) \geq \sum_{i=1}^{n} s_i$$

and $\sum_{i=1}^{n} k_i \leq m + n$. Since $(r^i, s_i) \in \operatorname{conv}_{k_i} C_i$, there exists $\tilde{x}^{ij} \in \mathbb{R}^{n_i}$, $\beta_{ij} \geq 0$ for $j = 1, \dots, k_i$ such that $f_i(\tilde{x}^{ij}) < +\infty$, $\sum_{j=1}^{k_i} \beta_{ij} = 1$, and

$$r^i = \sum_{j=1}^{k_i} \beta_{ij} A_i \tilde{x}^{ij}, \quad s_i = \sum_{j=1}^{k_i} \beta_{ij} f_i(\tilde{x}^{ij}).$$

[3]Here if we apply Theorem 2.3 with dimension $m + 1$ instead of using Corollary 2.7, the rest of the argument still works except that the bound for $\sum_{i=1}^{n} k_i$ has to be weakened to $m + n + 1$ from $m + n$. Therefore, the consideration of extremeness in Corollary 2.7 provides the exact improvement parallel to how (1.5) improves from the earliest bound (1.3).

Thus,

$$\kappa \geq \sum_{i=1}^{n}\sum_{j=1}^{k_i} \beta_{ij} A_i \tilde{x}^{ij} - b = \sum_{i=1}^{n} A_i \sum_{j=1}^{k_i} \beta_{ij}\tilde{x}^{ij} - b \tag{4.2}$$

and

$$\sum_{i=1}^{n} \rho_i^{k_i} + \eta \geq \sum_{i=1}^{n} \left(\rho_i^{k_i} + \sum_{j=1}^{k_i} \beta_{ij} f_i(\tilde{x}^{ij})\right) \geq \sum_{i=1}^{n} f_i \left(\sum_{j=1}^{k_i} \beta_{ij}\tilde{x}^{ij}\right). \tag{4.3}$$

From (4.2) we know that

$$\left(\sum_{j=1}^{k_1} \beta_{1j}\tilde{x}^{1j}, \ldots, \sum_{j=1}^{k_n} \beta_{nj}\tilde{x}^{nj}\right)$$

is feasible to the perturbed problem related to $v(\kappa)$, and so the corresponding objective value satisfies

$$\sum_{i=1}^{n} f_i \left(\sum_{j=1}^{k_i} \beta_{ij}\tilde{x}^{ij}\right) \geq v(\kappa).$$

The above inequality, together with (4.3) and (4.1), implies

$$v^{**}(0) + \epsilon + \sum_{i=1}^{n} \rho_i^{k_i} \geq v(0) - \epsilon.$$

We finish the proof by letting $\epsilon \to 0$. Because all the k_i depend on ϵ, we have to choose the worst case of $\sum_{i=1}^{n} \rho_i^{k_i}$ encountered in this process. $\square$

From a computational viewpoint, since we do not know the k_i that appeared in Theorem 4.1, in order to find a number for the bound, we have to find the worst case k_i by solving the following optimization problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \rho_i^{k_i} \\ \text{s. t.} \quad & 1 \leq k_i \leq m+1,\ k_i \in \mathbb{Z}, \quad \forall i = 1, \ldots, n, \\ & \sum_{i=1}^{n} k_i \leq m + n. \end{aligned} \tag{4.4}$$

Let B be the optimal value of (4.4). Then

$$B \leq \sum_{i=1}^{n} \rho(f_i).$$

On the other hand, since for any feasible solution of (4.4) the number of k_i with $k_i \geq 2$ is bounded by m, thus

$$B = \sum_{i:k_i \geq 2} \rho_i^{k_i} \leq \sum_{i=1}^{m} \rho(f_i)$$

if $\rho(f_1) \geq \cdots \geq \rho(f_n)$. The above argument shows that the bound B given by the optimization problem (4.4) is at least as tight as the bound (1.5) in [14].

To illustrate the procedure to calculate the bound B, consider the simple case where all the x^i in the primal problem (1.1) are one-dimensional and all the functions f_i are equal to the same function f. In this case, $\rho_i^{k_i} = \rho(f)$ if $k_i \geq 2$. The optimal value to (4.4) is attained when the number of k_i that equal 2 is maximized, so the optimal value is $\min\{m, n\}\rho(f)$, which is the same as the result given by (1.5). Example 1 used in [14] belongs to this category. It hence explains why the bound (1.5) is tight for that example. However, if the dimension of x^i in the primal problem can be arbitrarily large, the bound (1.5) can be very loose. As will be shown in section 5, the difference between the bound (1.5) and the exact duality gap tends to infinity for a series of problems.

5. Applications.

5.1. Joint routing and congestion control in networking. In this part, we will first apply the previous result to the network utility maximization problem. Consider a network with N users and L links. Let a strictly positive vector $c \in \mathbb{R}^L$ contain the capacity of each link. Each user i has K^i available paths on which to send its commodity. We assume that the users are sorted such that $K^1 \geq \cdots \geq K^N$. The routing matrix of user i, denoted by R^i, is an $L \times K^i$ matrix defined by

$$R^i_{lk} = \begin{cases} 1 & \text{if the } k\text{th path of user } i \text{ passes through link } l, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x^i \in \mathbb{R}^{K^i}$ be the vector in which x^i_k is the amount of commodity sent by user i on its kth path. Assume that each user i has a utility function $U_i(\cdot)$ depending on the vector x^i. Then the network utility maximization problem can be written as

$$\begin{aligned} \max \quad & \sum_{i=1}^N U_i(x^i) \\ \text{s.t.} \quad & \sum_{i=1}^N R^i x^i \leq c, \\ & x^i \geq 0 \quad \forall i = 1, \ldots, N. \end{aligned} \tag{5.1}$$

If all the utility functions $U^i(\cdot)$ are concave, then the above problem (5.1) can be solved by standard convex optimization techniques. Difficulty arises when $U^i(\cdot)$ is not concave. For example, if we restrict each user to choosing only one path (single-path routing) and want to maximize the total throughput of the network, then the corresponding utility function is

$$U_i(x^i) = \max_{s=1,\ldots,K^i} x^i_s.$$

Define

$$f_i(x^i) = \begin{cases} \min\limits_{s=1,\ldots,K^i} (-x^i_s) & \text{if } 0 \leq x^i \leq \|c\|_\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

Here $\|c\|_\infty$ is the maximum link capacity in the network. Now the original network utility maximization problem (5.1) is equivalent to the following problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} f_i(x^i) \\ \text{s.t.} \quad & \sum_{i=1}^{N} R^i x^i \le c. \end{aligned} \tag{5.3}$$

The above problem is a particular case of the general optimization problem with separable objectives (1.1) studied in this paper. Using the same technique as shown in Example 3.4, we can prove that

$$\rho^k(f_i) = \frac{k-1}{k}\|c\|_\infty, \quad k = 1, \dots, K^i,$$
$$\rho^{K^i+1}(f_i) = \rho(f_i) = \frac{K^i-1}{K^i}\|c\|_\infty.$$

In the following, suppose each user has a large number of paths to select. More explicitly, $K^i \ge L+1$ is assumed for user i. Based on the bound (1.5), the duality gap is bounded by

$$\sum_{i=1}^{\min\{N,L\}} \frac{K^i-1}{K^i}\|c\|_\infty,$$

which is at least

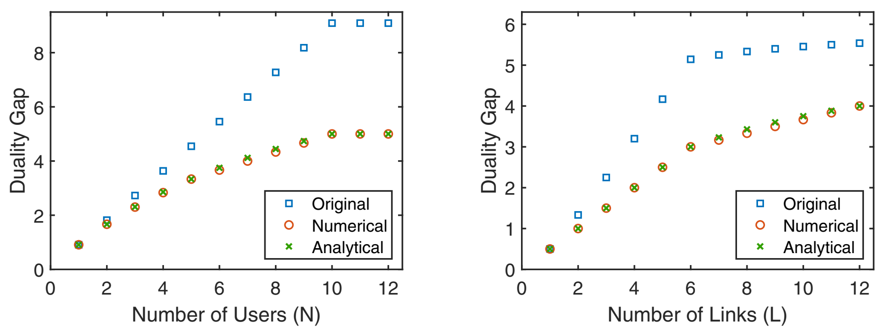
$$\min\{N, L\}\frac{L}{L+1}\|c\|_\infty. \tag{5.4}$$

In contrast, by Theorem 4.1, the duality gap is bounded by the optimal value of the following optimization problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} \frac{k_i-1}{k_i}\|c\|_\infty \\ \text{s.t.} \quad & 1 \le k_i \le L+1,\ k_i \in \mathbb{Z}, \quad \forall i = 1, \dots, N, \\ & \sum_{i=1}^{N} k_i \le N + L. \end{aligned} \tag{5.5}$$

Let N' be the number of users whose $k_i \ge 2$. Then $0 \le N' \le \min\{N, L\}$. If $N' > 0$, using the inequality between arithmetic mean and harmonic mean,

$$\begin{aligned} \sum_{i=1}^{N} \frac{k_i-1}{k_i} &= \sum_{i:k_i\ge 2} \frac{k_i-1}{k_i} = N' - \sum_{i:k_i\ge 2}\frac{1}{k_i} \\ &\le N' - \frac{N'^2}{\sum_{i:k_i\ge 2} k_i} \le N' - \frac{N'^2}{N'+L} \\ &= \frac{L}{1+L/N'} \le \min\{N,L\}\frac{L}{L+\min\{N,L\}}. \end{aligned}$$



(a) $L = 10$ (b) $N = 6$

FIG. 2. *The comparison among the original bound* (5.4)*, the numerical result from directly solving* (5.5)*, and the analytical result for the linear utility case.*

The above analysis provides an upper bound for problem (5.5), which in turn is an upper bound for the duality gap. Taking the $N \geq L$ case as an example, by the above inequality, we can bound the duality gap by $L\|c\|_\infty/2$, essentially half of the bound given by (1.5). The same result was obtained by a specialized technique in [7].

In principle, we can directly solve (5.5) to yield better upper bounds, and this is particularly practical for small instances. In Figure 2, we compare the original bound (5.4), the numerical result from directly solving (5.5), and the above analytical result for fixed N or fixed L with the assumption that $\|c\|_\infty = 1$. Figure 2 shows that our analytical result is much better than the original bound, and in all cases it is almost tight compared with the numerical result. In fact, the above analysis can be regarded as solving problem (5.5) exactly without considering all integer constraints on k_i.

Next, we consider another case in which each user has logarithmic utility but still must choose only one path. The utility function of user i can be written as

$$U_i(x^i) = \log \max_{s=1,\ldots,K^i} x_s^i.$$

Define

$$g_i(x^i) = \begin{cases} -\log \max\limits_{s=1,\ldots,K^i} x_s^i & \text{if } 0 \leq x^i \leq \|c\|_\infty,\ x^i \neq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the network utility maximization problem (5.1) is equivalent to the problem obtained by replacing f_i with g_i in (5.3). Using the result in Example 3.5,

$$\rho^k(g_i) = \log k, \quad k = 1, \ldots, K^i,$$
$$\rho^{K^i+1}(g_i) = \rho(g_i) = \log K^i.$$

Applying the bound (1.5) to this case, we can bound the duality gap by

$$\sum_{i=1}^{\min\{N,L\}} \log K^i, \tag{5.6}$$

which is at least $\min\{N, L\} \log(L+1)$. On the other hand, by Theorem 4.1, the duality gap is bounded by the optimal value of the following optimization problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} \log k_i \\ \text{s.t.} \quad & 1 \le k_i \le L+1,\ k_i \in \mathbb{Z}, \quad \forall i = 1, \dots, N, \\ & \sum_{i=1}^{N} k_i \le N + L. \end{aligned} \tag{5.7}$$

If we still let N' be the number of users whose $k_i \ge 2$, then $0 \le N' \le \min\{N, L\}$ and the above bound

$$\begin{aligned} \sum_{i=1}^{N} \log k_i = \sum_{i:k_i \ge 2} \log k_i = \log \prod_{i:k_i \ge 2} k_i \le \log \left(\frac{\sum_{i:k_i\ge 2} k_i}{N'} \right)^{N'} \\ \le \log \left(\frac{N'+L}{N'} \right)^{N'} \le \min\{N, L\} \log \left(1 + \frac{L}{\min\{N, L\}} \right), \end{aligned}$$

where in the last step the monotonicity of the function $(1+1/x)^x$ is used. Note that the new bound is qualitatively tighter than the bound (5.6) provided by (1.5) by removing a logarithm factor of $O(\log L)$ when $N \ge L$.

5.2. Dynamic spectrum management in communication. Consider a communication system consisting of L users sharing a common band. The band is divided equally into N tones. Each user l has a power budget p_l which can be allocated across all the tones. Let x_l^i be the power of user l allocated on tone i. Due to the crosstalk interference between users, the total noise for a user on tone i is the sum of a background noise σ_i and the power of all other users on the same tone. Therefore, the achievable transmission rate of user l on tone i is given by

$$u_l^i = \frac{1}{N} \log \left(1 + \frac{x_l^i}{\|x^i\|_1 - x_l^i + \sigma_i} \right).$$

The dynamic spectrum management problem is to maximize the total throughput of all users under the power budget constraints, which can be formulated as the following nonconvex optimization problem:

$$\begin{aligned} \max \quad & \sum_{l=1}^{L} \sum_{i=1}^{N} u_l^i \\ \text{s.t.} \quad & \sum_{i=1}^{N} x_l^i \le p_l \quad \forall l = 1, \dots, L, \\ & x_l^i \ge 0 \quad \forall i = 1, \dots, N,\ \forall l = 1, \dots, L. \end{aligned} \tag{5.8}$$

For simplicity, we assume that the noises $\sigma_i \le 1$ and the power budgets $p_l \le 1$ (if not, then scale all the σ_i and p_l simultaneously). The latter requires all the variables $x_l^i \le 1$. Using the function h_σ introduced in Example 3.6, the objective function of (5.8) can be rewritten as a sum of separable objectives:

$$\sum_{l=1}^{L} \sum_{i=1}^{N} u_l^i = -\frac{1}{N} \sum_{i=1}^{N} h_{\sigma_i}(x^i).$$

For the purpose of designing dual algorithms, it is of great interest to estimate the duality gap for problem (5.8). In [15], the authors showed that the duality gap will tend to zero if the number of users L is fixed and the number of tones N goes to infinity. The paper [12] further determined the convergence rate of the duality gap to be $O(1/\sqrt{N})$. Using the bound (1.5), we now demonstrate how to improve the convergence rate estimation to $O(1/N)$, which can only be achieved by the method in [12] in the special case where all the noises σ_i are the same.[4]

Example 3.6 proves that the nonconvexity

$$\rho^k(h_{\sigma_i}) \le \log \frac{k}{\sigma_i} \le \log \frac{k}{\sigma}, \quad k = 1, \dots, L+1,$$

where σ is the minimum among all the noises σ_i, so (1.5) implies that the duality gap is upper bounded by

$$\frac{\min\{N, L\}}{N} \log \frac{L+1}{\sigma}, \tag{5.9}$$

which is in the order of $O(1/N)$ if L is fixed and N increases.

In order to further improve the estimation (5.9) for the duality gap, we can resort to Theorem 4.1 and follow the exact same steps for solving (5.7), which shows that the duality gap is upper bounded by

$$\frac{\min\{N, L\}}{N} \log \frac{1 + L/\min\{N, L\}}{\sigma}.$$

Like the previous example, our bound is still tighter than the one (5.9) from (1.5) by removing a logarithm factor.

6. Generalization with nonlinear constraints. The idea in this paper can also be applied to separable problems with nonlinear constraints such as

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} f_i(x^i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} g_i(x^i) \le b. \end{aligned} \tag{6.1}$$

Here each f_i has the same requirement as in (1.1), and each $g_i : \mathbb{R}^{n_i} \to \bar{\mathbb{R}}^m$ is proper and lower semicontinuous. Note that the previous problem (1.1) we studied is a special case of the optimization problem (6.1) if we choose $g_i(x^i) = A_i x^i$. Let $y \in \mathbb{R}^m_+$ be the dual variables. Then the Lagrangian is

$$L(x, y) = \sum_{i=1}^{n} (f_i(x^i) + y^T g_i(x^i)) - y^T b$$

and the Lagrange dual problem of (6.1) is

$$d = \sup_{y \ge 0} \inf_{x} L(x, y).$$

[4]The paper [12] actually studied the generalization of problem (5.8) under the existence of a path loss coefficient between different users. However, the argument for $O(1/N)$ provided here can also be adapted to the general problem.

If the functions g_i are not convex, the duality gap should not only depend on the nonconvexity of functions f_i but also somehow relate to the functions g_i. Like [6], we define the kth order nonconvexity of a proper function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ with respect to another proper function $g : \mathbb{R}^n \to \bar{\mathbb{R}}^m$, denoted by $\rho^k(f, g)$. To do this, for each $x \in \mathbb{R}^n$, we introduce a set $G^k(x; g) \subseteq \mathbb{R}^m$ such that $y \in G^k(x; g)$ if and only if there exist $x^j \in \mathbb{R}^n$ and $\beta_j \in \mathbb{R}$, for $j = 1, \ldots, k$, satisfying

$$\begin{cases} x = \displaystyle\sum_{j=1}^{k} \beta_j x^j, \\ y = \displaystyle\sum_{j=1}^{k} \beta_j g(x^j), \quad g(x^j) < +\infty,\ \forall j = 1, \ldots, k, \\ \displaystyle\sum_{j=1}^{k} \beta_j = 1, \quad \beta_j \geq 0,\ \forall j = 1, \ldots, k. \end{cases} \tag{6.2}$$

Define the auxiliary function $h^k(x; f, g) : \mathbb{R}^n \to (-\infty, +\infty]$ by

$$h^k(x; f, g) = \inf_{z \in \mathbb{R}^n} \{ f(z) | g(z) \leq y\ \forall y \in G^k(x; g) \}. \tag{6.3}$$

Then $\rho^k(f, g)$ is defined by

$$\rho^k(f, g) = \sup \left\{ h^k \left(\sum_{j=1}^{k} \alpha_j x^j; f, g \right) - \sum_{j=1}^{k} \alpha_j f(x^j) \right\}$$

over all possible convex combinations $\alpha_j \geq 0$, $j = 1, \ldots, k$, with $\sum_{j=1}^{k} \alpha_j = 1$ of points x^j satisfying $f(x^j) < +\infty$. If the function g is convex, then in the infimum of (6.3) we can choose $z = x$, which gives $h^k(x; f, g) \leq f(x)$ and $\rho^k(f, g) \leq \rho^k(f)$. However, if g is not convex, the above argument does not work since in the worst case $g(x)$ could be $+\infty$ and does not satisfy the constraint in the infimum of (6.3).

Theorem 4.1 can be modified accordingly to the case with nonlinear constraints by replacing $\rho^{k_i}(f_i)$ with $\rho^{k_i}(f_i, g_i)$. In the case when all g_i are convex, $\rho^{k_i}(f_i, g_i) \leq \rho^{k_i}(f_i)$, which implies that the original conclusion in Theorem 4.1 remains true. However, in general, considering g_i explicitly in Theorem 6.1 has the potential to provide a tighter bound even for convex constraints including the linear cases.

THEOREM 6.1. *Assume that the primal problem* (6.1) *is feasible, i.e.,* $p < +\infty$. *Then there exist integers* $1 \leq k_i \leq m + 1$ *such that* $\sum_{i=1}^{n} k_i \leq m + n$ *and the duality gap*

$$p - d \leq \sum_{i=1}^{n} \rho_i^{k_i}.$$

Here $\rho_i^k = \rho^k(f_i, g_i)$ *is the* k*th nonconvexity of function* f_i *with respect to function* g_i.

Proof. See Appendix B. ∎

In the following, we demonstrate how to use the above theory to estimate the duality gap for a separable problem with both nonconvex objective and nonconvex constraints. Consider a modification of the optimization problem that appeared in the network utility maximization in section 5.1 but with additional nonconvex constraints:

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} f_i(x^i) \\ \text{s. t.} \quad & \sum_{i=1}^{N} R^i x^i \leq c, \\ & 0 \leq x^i \leq a \text{ or } b \leq x^i \leq \|c\|_\infty \quad \forall i = 1, \ldots, N. \end{aligned} \tag{6.4}$$

Here $f_i(x^i)$ is defined in (5.2) and $a, b \in \mathbb{R}$ are constants satisfying $0 < a < b < \|c\|_\infty$. We define the function g_i to capture the constraints:

$$g_i(x^i) = \begin{cases} R^i x^i & \text{if } 0 \leq x^i \leq a \text{ or } b \leq x^i \leq \|c\|_\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that in this problem all the matrices R^i are nonnegative. Therefore, for given $x^i \in \mathbb{R}^{K^i}$ with $0 \leq x^i \leq \|c\|_\infty$, if we find a point $z \in \mathbb{R}^{K^i}$ with the property that $z \leq x^i$ and $0 \leq z \leq a$ or $b \leq z \leq \|c\|_\infty$, then the constraint in the infimum of (6.3) will be automatically satisfied. One possible choice for such a point z is given by

$$z_s = \begin{cases} x_s^i & \text{if } 0 \leq x_s^i \leq a \text{ or } b \leq x_s^i \leq \|c\|_\infty, \\ a & \text{otherwise,} \end{cases}$$

which satisfies $f_i(z) - f_i(x^i) \leq b - a$. By the definition of the nonconvexity, we have

$$h^k(x^i; f_i, g_i) \leq f_i(x^i) + b - a$$

and

$$\begin{aligned} \rho^k(f_i, g_i) &\leq \sup \left\{ f_i\left(\sum_{j=1}^{k} \alpha_j x^{ij} \right) - \sum_{j=1}^{k} \alpha_j f_i(x^{ij}) \right\} + b - a \\ &\leq \rho^k(f_i) + b - a. \end{aligned}$$

Now by the same technique used to solve (5.5) in section 5.1, we can show that the duality gap is bounded by

$$\min\{N, L\} \frac{L}{L + \min\{N, L\}} + L(b - a).$$

Compared with the result obtained in section 5.1, the extra term $L(b - a)$ is the nonlinear constraints' contribution, which is zero when $a = b$, as expected.

7. Conclusion. The improvements obtained in this paper are attributed to two sources. First, instead of using a single number measurement, a series of numbers are introduced to characterize the nonconvexity of a function in a potentially much finer manner. This is based on the concept of the kth convex hull of a set, which allows us to differentiate different levels of nonconvexity for nonconvex sets. Second, for a separable nonconvex problem, instead of approximating each subproblem individually, we consider all of them jointly. Based on the fact that the total deviation of each subproblem to a convex problem is bounded, a much tighter duality gap estimation can be reached.

In this paper, we focus on estimating the duality gap without consideration of actually solving the primal problem (1.1). A natural future direction is to design approximate algorithms for the primal problem and analyze the quality of the obtained solution based on our deeper understanding of the nonconvexity achieved in this paper.

Appendix A. The nonconvexity of the capacity function. In this appendix, we are going to compute the nonconvexity of the capacity function $h_\sigma(x)$, defined in Example 3.6, which appears in the dynamic spectrum management problem in section 5.2. Define an auxiliary function

$$H(x;\sigma) = \prod_{s=1}^{n} \frac{\|x\|_1 - x_s + \sigma}{\|x\|_1 + \sigma}.$$

Then $h_\sigma(x) = \log H(x;\sigma)$. To compute the kth nonconvexity for the function h_σ, we first prove some elementary properties for the function $H(x;\sigma)$.

Lemma A.1. *The function $H(x;\sigma)$ has the following properties:*

(a) *For any vectors x and y in the region $0 \le x, y \le \sigma$, if $y \le x$, then $H(y;\sigma) \ge H(x;\sigma)$.*

(b) $\sigma H(x;1) \le H(x;\sigma) \le H(x;1)$.

Proof. For any x in the region $0 \le x \le \sigma$, the partial derivatives

$$\begin{aligned}\frac{\partial H(x;\sigma)}{\partial x_i} &= H(x;\sigma)\left(\sum_{s=1}^{n} \frac{1}{\|x\|_1 - x_s + \sigma} - \frac{1}{\|x\|_1 - x_i + \sigma} - \frac{n}{\|x\|_1 + \sigma}\right)\\ &= H(x;\sigma)\left(\sum_{s=1}^{n} \frac{x_s}{(\|x\|_1 - x_s + \sigma)(\|x\|_1 + \sigma)} - \frac{1}{\|x\|_1 - x_i + \sigma}\right)\\ &\le H(x;\sigma)\left(\sum_{s=1}^{n} \frac{x_s}{\|x\|_1(\|x\|_1 + \sigma)} - \frac{1}{\|x\|_1 + \sigma}\right) = 0,\end{aligned}$$

which gives the first property.

For the second property, it is obvious to see that $H(x;\sigma) \le H(x;1)$. The other inequality is equivalent to

$$p(\sigma) = \frac{1}{\sigma} H(x;\sigma) - H(x;1) \ge 0.$$

The partial derivative

$$\begin{aligned}\frac{\partial H(x;\sigma)}{\partial \sigma} &= H(x;\sigma)\left(\sum_{s=1}^{n} \frac{1}{\|x\|_1 - x_s + \sigma} - \frac{n}{\|x\|_1 + \sigma}\right)\\ &= H(x;\sigma)\sum_{s=1}^{n} \frac{x_s}{(\|x\|_1 - x_s + \sigma)(\|x\|_1 + \sigma)}\\ &\le H(x;\sigma)\sum_{s=1}^{n} \frac{x_s}{\sigma(\|x\|_1 + \sigma)}\\ &= \frac{1}{\sigma} H(x;\sigma)\frac{\|x\|_1}{\|x\|_1 + \sigma} \le \frac{1}{\sigma} H(x;\sigma)\end{aligned}$$

implies that

$$p'(\sigma) = -\frac{1}{\sigma^2} H(x;\sigma) + \frac{1}{\sigma} \frac{\partial H(x;\sigma)}{\partial \sigma} \leq 0.$$

Therefore, the function $p(\sigma)$ is nonincreasing. Together with $p(1) = 0$, we have proved the nonnegativity of $p(\sigma)$. ∎

To upper bound the kth nonconvexity of the function h_σ, consider arbitrary points x^j for $j = 1, \ldots, k$ with corresponding combination weights $\alpha_j > 0$. Define k vectors $y^1, \ldots, y^k$ in $\mathbb{R}^k$ by

$$\begin{aligned} y^1 &= (1/H(x^1;1), 0, \ldots, 0),\\ y^2 &= (0, 1/H(x^2;1), \ldots, 0),\\ &\ldots,\\ y^k &= (0, 0, \ldots, 1/H(x^k;1)). \end{aligned}$$

Using the result of nonconvexity for the function g given in Example 3.5 and the properties proved in Lemma A.1, we have

$$\begin{aligned} h_\sigma\left(\sum_{j=1}^k \alpha_j x^j\right) &= \log H\left(\sum_{j=1}^k \alpha_j x^j; \sigma\right) \leq \log H\left(\sum_{j=1}^k \alpha_j x^j; 1\right) && \text{by property (b)}\\ &\leq \log H(\alpha_j x^j; 1) \quad \forall j = 1, \ldots, k, && \text{by property (a)} \end{aligned}$$

and then

$$\begin{aligned} & h_\sigma\left(\sum_{j=1}^k \alpha_j x^j\right) \leq \log \min_{j=1,\ldots,k} H(\alpha_j x^j; 1)\\ &\leq \log \min_{j=1,\ldots,k} \frac{1}{\alpha_j} H(\alpha_j x^j; \alpha_j) = \log \min_{j=1,\ldots,k} \frac{1}{\alpha_j} H(x^j; 1) && \text{by property (b)}\\ &= g\left(\sum_{j=1}^k \alpha_j y^j\right) \leq \sum_{j=1}^k \alpha_j g(y^j) + \log k && \text{by the nonconvexity of } g\\ &= \sum_{j=1}^k \alpha_j \log H(x^j; 1) + \log k \leq \sum_{j=1}^k \alpha_j \log \frac{1}{\sigma} H(x^j; \sigma) + \log k && \text{by property (b)}\\ &= \sum_{j=1}^k \alpha_j h_\sigma(x^j) + \log \frac{k}{\sigma}. \end{aligned}$$

The above argument shows that the kth nonconvexity $\rho^k(h_\sigma) \leq \log(k/\sigma)$.

In the above example, an upper bound for the kth nonconvexity of function h_σ is obtained by a reduction from the nonconvexity of g in Example 3.5. Along this line of thought, it is conceivable to find the exact value for the kth nonconvexity of h_σ if we are able to reduce h_σ to itself (but with just k variables).

Appendix B. Proof of Theorem 6.1. In this appendix, we give the complete proof for the bound of the duality gap for problem (6.1) with separable objectives and separable but possibly nonlinear constraints.

Proof of Theorem 6.1. Like the proof of Theorem 4.1, we define the perturbation function $v : \mathbb{R}^m \to \bar{\mathbb{R}}$ by letting $v(z)$ be the optimal value of the perturbed problem

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} f_i(x^i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} g_i(x^i) \le b + z. \end{aligned}$$

The optimal value of the above problem can always be achieved because the constraint set is closed, which is implied by the lower-semicontinuity of g_i.

By the same argument as in the proof of Theorem 4.1, $-\infty < v^{**}(0) = d \le v(0) = p < +\infty$, and for $\epsilon > 0$ there exists $(\kappa, \eta) \in \operatorname{conv} \operatorname{epi} v$ which is sufficiently close to $(0, v^{**}(0))$ such that

$$v(\kappa) \ge v(0) - \epsilon, \quad \eta \le v^{**}(0) + \epsilon.$$

Proceed exactly the same as in the proof of Theorem 4.1. We decompose κ into z^j such that

$$\kappa = \sum_{j=1}^{l} \alpha_j z^j, \quad \eta \ge \sum_{j=1}^{l} \alpha_j v(z^j)$$

and introduce $w^j \in \mathbb{R}^m_+$ with

$$\left(b + \kappa - \sum_{j=1}^{l} \alpha_j w^j, \sum_{j=1}^{l} \alpha_j v(z^j) \right) \in \operatorname{conv} \sum_{i=1}^{n} C_i,$$

where C_i is defined by

$$C_i = \{(g_i(x^i), f_i(x^i)) | f_i(x^i) < +\infty, g_i(x^i) < +\infty, x^i \in \mathbb{R}^{n_i}\}.$$

Corollary 2.7 gives points $(r^i, s_i) \in \operatorname{conv}_{k_i} C_i$ with $1 \le k_i \le m+1$ such that

$$b + \kappa \ge b + \kappa - \sum_{j=1}^{l} \alpha_j w^j = \sum_{i=1}^{n} r^i, \quad \eta \ge \sum_{j=1}^{l} \alpha_j v(z^j) \ge \sum_{i=1}^{n} s_i,$$

and $\sum_{i=1}^{n} k_i \le m + n$. Since $(r^i, s_i) \in \operatorname{conv}_{k_i} C_i$, there exist $\tilde{x}^{ij} \in \mathbb{R}^{n_i}$, $\beta_{ij} \ge 0$ for $j = 1, \dots, k_i$ such that $f_i(\tilde{x}^{ij}) < +\infty$, $g_i(\tilde{x}^{ij}) < +\infty$, $\sum_{j=1}^{k_i} \beta_{ij} = 1$, and

$$r^i = \sum_{j=1}^{k_i} \beta_{ij} g_i(\tilde{x}^{ij}), \quad s_i = \sum_{j=1}^{k_i} \beta_{ij} f_i(\tilde{x}^{ij}).$$

For each $i = 1, \dots, n$, define $\hat{x}^i = \sum_{j=1}^{k_i} \beta_{ij} \tilde{x}^{ij}$. If $h^{k_i}(\hat{x}^i; f_i, g_i) = +\infty$, we also have $\rho_i^{k_i} = +\infty$ and the theorem is trivial in this case. Otherwise, observe that

$$\sum_{j=1}^{k_i} \beta_{ij} g_i(\tilde{x}^{ij}) \in G^{k_i}(\hat{x}^i; g_i),$$

because $\tilde{x}^{ij}$ and β_{ij} satisfy all the constraints given in (6.2). As a result, there will be $\hat{q}^i \in \mathbb{R}^{n_i}$ such that

$$g_i(\hat{q}^i) \le \sum_{j=1}^{k_i} \beta_{ij} g_i(\tilde{x}^{ij}),$$
$$f_i(\hat{q}^i) \le h^{k_i}(\hat{x}^i; f_i, g_i) + \epsilon.$$

Thus,

$$\kappa \ge \sum_{i=1}^{n} \sum_{j=1}^{k_i} \beta_{ij} g_i(\tilde{x}^{ij}) - b \ge \sum_{i=1}^{n} g_i(\hat{q}^i) - b$$

and

$$\sum_{i=1}^{n} \rho_i^{k_i} + \eta \ge \sum_{i=1}^{n} \left(\rho_i^{k_i} + \sum_{j=1}^{k_i} \beta_{ij} f_i(\tilde{x}^{ij}) \right) \ge \sum_{i=1}^{n} h_i^{k_i}(\hat{x}^i; f_i, g_i) \ge \sum_{i=1}^{n} f_i(\hat{q}^i) - n\epsilon.$$

Now $(\hat{q}^1, \ldots, \hat{q}^n)$ is a feasible solution to the perturbed problem $v(\kappa)$. Following the original proof, we have

$$v^{**}(0) + \epsilon + \sum_{i=1}^{n} \rho_i^{k_i} \ge v(0) - (n+1)\epsilon$$

and then finish the proof by letting $\epsilon \to 0$. ∎

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