Communication Systems with Partially Existing Encoder and Decoder

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***Abstract*—Design of the optimal encoder and decoder for a discrete memoryless channel is a well-studied problem in classic information theory. However, there exists important systems which operate with components (encoder and decoders) that are prebuilt and possibly mismatched, limiting the achievable performance metrics. In this paper, we propose a model that takes into account such constraints. Using this model, we develop an understanding of the source coding problem in this context and introduce two new functions -transitory and restricted rate distortion- and analyze certain properties of them. In addition, one important application of this model for developing a comprehensive theory of semantic communication is discussed. Finally, some numerical examples are provided to illustrate the concepts mentioned in the paper.**

I. INTRODUCTION

Classic communication systems can be conceptualized as the model in Figure 1 [1]. There exists a source that produces a sequence of messages that it intends to transmit to the destination via a noisy channel. In order to optimize the communication, an encoder-decoder pair is designed, which encodes the source sequences before transmitting and decodes the noisy codeword received by the destination into a sequence.

The optimal encoder/decoder design for a discrete memoryless channel has been studied extensively, and Shannon's "source-channel coding theorem" states that for both the lossy and lossless cases, separately designing the source encoder (compression) and channel encoder (error protection) is optimal. However, Shannon's theorem and results stemming from it assume full control over the joint design of this pair. This assumption is not always valid. For example, imagine a case where the source encoder and decoder are predesigned and built into the source and destination (and probably mismatched). As a result, the channel encoder/decoder has to be designed in a way to compensate for both the noise of the channel and the suboptimality of the source encoder. In this paper, we aim to build an understanding of such systems.

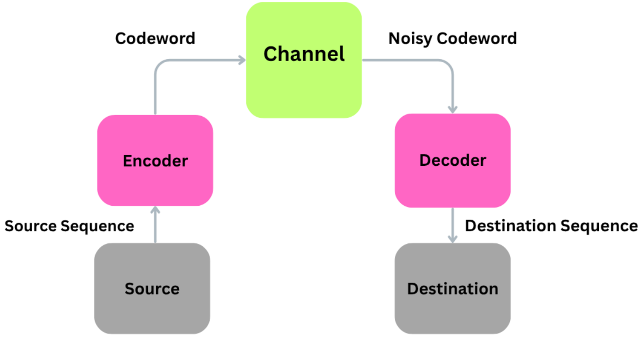

Fig. 1. Conceptual model of a communication system

One major potential application of this paper is making steps towards a comprehensive theory of semantic communication. In a traditional communication framework (Shannon's Information Theory), the main question regarding the quality of communication is the technical problem of "How accurately can the symbols of communication be transmitted?". In reality, in addition to this, there is another level of problem that arises in communication called the semantic problem. The semantic problem asks the fundamental question about the accuracy of transmitted symbols (syntax) conveying the desired meaning to the receiver (semantics). Shannon argues that the semantic aspects of communication are irrelevant to the engineering problem. However, semantic communication in recent years has gained attention due to applications in deep learning, federated learning, and end-to-end communication for IoT devices. These works include, but are not limited to [2], [3], [4], [5].

Attempts at a theory of semantic communication have been made. Classical results use logical probability and truth values [6], [7], [8], and [9]. Modern works also view semantic communication through game theory by [10], introduction of synonymous mapping by [11], and characterization of achievable pairs of semantic distortion and sematic cost by [12]. However, all these works lack the crucial insight and complexity of semantic communication, which arises from the difference in world models of the two parties communicating. We will show that our model can be a guideline towards capturing the essence

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of semantic misunderstanding.

In this paper, we provide a simple model for communication that happens with some unchangeable components (other than the channel, which is unchangeable in the classical model, too). We then explore the optimal design for the case where the channel is perfect. Two novel rate distortion functions (namely, transitory and restricted) are introduced. Important properties of these functions are proven. Finally, the connection to semantic communication is explored.

The rest of the paper is organized as follows. The model is defined in section II. We explore the perfect channel case in section III, and the equivalents of rate-distortion functions are presented. The connection to semantic communication is discussed in section IV. Some numerical results are presented in V. Section VI concludes the paper.

II. MODEL

In this section, we will formally define the model.

A. General Framework

We first provide our general model without assuming any inherent constraints on any of the components. Similar to the classic information theory case, we have a source that produces sequences according to a prior distribution.

These sequences then pass through a prebuilt encoder. This encoder will encode the sequences into ”transitory codewords”. Before sending these codewords through the channel, we can design an encoder to encode the transitory codewords into ”actual codewords” that will serve as equivalent to the codewords in the traditional communication system.

These encoded codewords will then pass through a channel. Analogous to the encoders on the source side, we have decoders on the destination side. This system is summarized in Figure 2.

B. Symbol by Symbol Case

Now, we will provide a special case of the model proposed in the previous section that we will use for the remainder of our paper.

Let the set of all possible sequences be $\mathcal{S}$. $\mathcal{S}$ is a countable finite set. The source will produce a sequence s with probability $p(s)$ (the prior distribution).

We will assume that the prebuilt encoder is a symbol to symbol encoder that will encode a sequence into a ”transitory codeword” that comes from the set $\mathcal{T}$. This encoding happens according to the transition probability q. In other words, the sequence s will be encoded into the transitory codeword t with probability $q(t|s)$. The assumption of symbol by symbol encoding ensures that the system preserves the iid and memoryless assumption that simplifies the analysis.

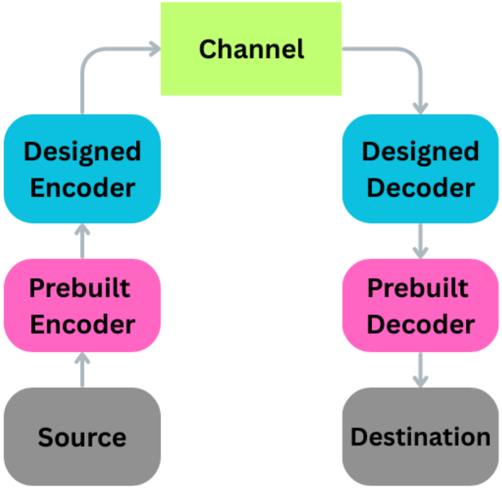

Fig. 2. Proposed model for communication

Similarly, for the designed encoder, let the set of all actual codewords be $\mathcal{M}$. The designed encoder will have the transition probability r. Thus, the transitory codeword t will be encoded as an actual codeword m with probability $r(m|t)$.

These encoded codewords will then pass through a channel with a transition probability c. Thus, a noisy codeword $\hat{m}$ will be received when codeword m is sent with probability $c(\hat{m}|m)$.

We will assume that the channel is discrete and memoryless. In other words, if the set of possible input symbols (we assume the output symbols are the same as input symbols for simplicity) is $\mathcal{X}$, then if $m = (x_1, x_2, \cdots, x_n)$ and $\hat{m} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$, we have:

$$c(\hat{m}|m) = \prod_{i=1}^{n} c(\hat{x}_i|x_i) \tag{1}$$

We show the cost of sending the codeword m with $l(m)$. Again, for simplicity, we will assume that transmitting each symbol will incur a cost equal to one. In other words, $l(m) = |m|$.

Analogous to the encoders on the source side, we have decoders on the destination side. We will define their transition probabilities with $\hat{q}$ and $\hat{r}$, for prebuilt and designed decoders, respectively.

We also define $d(s, \hat{s})$ to be a measure of distortion between source sequences s and $\hat{s}$. This function satisfies two properties:

$$\begin{aligned} d(s,s) &= 0, \quad \forall s \in \mathcal{S} \\ d(s,\hat{s}) &\geq 0, \quad \forall s, \hat{s} \in \mathcal{S} \end{aligned}$$

The goal is to design an encoder/decoder pair that minimizes the expected distortion between the source sequence and the sequence received, while keeping the expected cost of transmitting a source sequence less than

L. More precisely, we are trying to solve the following optimization problem

$$\begin{aligned} &\min_{r,\hat{r}} \mathbf{E}[d(s,\hat{s})] \\ &s.t. \quad E[l(m)] \leq L \end{aligned} \tag{2}$$

Equivalently, one can optimize the cost of transmitting while keeping the expected distortion less than a certain threshold D.

$$\begin{aligned} &\min_{r,\hat{r}} E[l(m)] \\ &s.t. \quad \mathbf{E}[d(s,\hat{s})] \leq D \end{aligned} \tag{3}$$

Remark. *Throughout this paper, we will use capital letters to show the transition matrices analogous to the transition probability functions when necessary. For instance, Q shows the transition matrix of the prebuilt encoder.*

Remark. *Note that conceptually speaking, we can have a case where the prebuilt encoder/decoder is the one on the channel side (switch the pink and blue components in Figure 2). But in this case, we can look at the combination of the prebuilt encoder/decoder and the channel as one imaginary channel.*

III. THE PERFECT CHANNEL CASE

In this section, we will assume that the channel is a lossless channel. We do this to focus on understanding the novelty of the new model. In doing so, we derive two new rate-distortion functions for this model.

A. Source Coding Theorem

First, we provide a quick review of the classic rate-distortion theorem. We are following the development in [13]. Assume we have a source producing sequences from $\mathcal{S}$ using the probability function $p(s)$. Let $\mathbf{S}$ be the random variable of the source sequence chosen. For any other random variable $\hat{\mathbf{S}}$ over the same set of sequences, we can define $I(S;\hat{S})$ to be the mutual information between $\mathbf{S}$ and $\hat{\mathbf{S}}$. In addition, the expected distortion is defined as:

$$E[d(\mathbf{S},\hat{\mathbf{S}})] = \sum_{s,\hat{s}\in\mathcal{S}} d(s,\hat{s})p(s,\hat{s})$$

Now, the rate distortion function is defined as

$$R(\delta) = \min_{\hat{\mathbf{S}}}\{I(\mathbf{S};\hat{\mathbf{S}}) : E[d(\mathbf{S},\hat{\mathbf{S}})] \leq \delta\} \tag{4}$$

Note that in [13], it starts the definition $R_k(\delta)$, which is for a series of sequences of length k. But as it proves $R_k(\delta) = R_1(\delta)$, we omit those steps for the sake of succinctness.

Remark. *One can view $\hat{\mathbf{S}}$ as a result of the sequence $\mathbf{S}$ passing a **test channel** with transition probability $p(\hat{s}|s)$. So the minimization is done over all test channels that ensure a distortion of less than δ.*

Now, using this function, the source coding theorem is presented as follows:

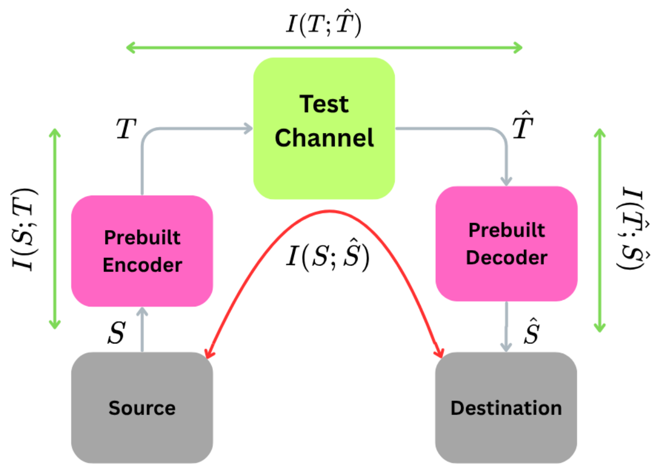

Fig. 3. Transtory Rate Distortion Test Channel

Theorem 1. *Fix a $\delta > \delta_{min}$ (where δ_{min} is the minimum achievable distortion. Then for any $\delta' > \delta$ and $R' > R(\delta)$, for sufficiently large k there exists a source code C of length k with M codewords where,*

1) $M \leq 2^{\lfloor kR' \rfloor}$
2) $d(C) < \delta'$

This theorem shows that we can find a compression ratio less than or equal to R' such that the resulting distortion is smaller than δ'. Now, we will see how this result can be extended to our model.

B. Source Coding Theorem for Transitory Codewords

The goal of this section is to see how much compression is achievable for transitory codewords while keeping the expected distortion below a certain level. Thus, it is natural to plug in the **test channel** between the prebuilt encoder and the prebuilt decoder. Figure 3 illustrates this point. We define the **transitory rate distortion** function for prebuilt encoders q and decoder $\hat{q}$ as

$$\bar{R}_{q,\hat{q}}(\delta) = \min_{\hat{\mathbf{T}}}\{I(\mathbf{T};\hat{\mathbf{T}}) : E[d(\mathbf{S},\hat{\mathbf{S}})] \leq \delta\} \tag{5}$$

Note that the minimum is taken over $I(\mathbf{T};\hat{\mathbf{T}})$. Before continuing, we prove a few properties of the transitory rate distortion function.

Lemma 1. $\forall q,\hat{q},\delta : R(\delta) \leq \bar{R}_{q,\hat{q}}(\delta)$

Proof. Let $\hat{\mathbf{T}}^*$ and $\hat{\mathbf{S}}^*$ be the random variables resulting from plugging in the optimal test channel in Figure 3. So $\bar{R}_{q,\hat{q}}(\delta) = I(\mathbf{T};\hat{\mathbf{T}}^*)$. Also note that $\mathbf{S} \rightarrow \mathbf{T} \rightarrow \hat{\mathbf{T}} \rightarrow \hat{\mathbf{S}}$ is a Markov chain. Thus, by the famous property of Markov chains $I(\mathbf{T};\hat{\mathbf{T}}^*) \geq I(\mathbf{S};\hat{\mathbf{S}}^*)$. Finally, as $E[d(\mathbf{S},\hat{\mathbf{S}}^*)] \leq \delta$, we have:

$$R(\delta) = \min_{\hat{\mathbf{S}}}\{I(\mathbf{S};\hat{\mathbf{S}}) : E[d(\mathbf{S},\hat{\mathbf{S}})] \leq \delta\} \leq I(\mathbf{S};\hat{\mathbf{S}}^*)$$

Combining all these results, we have:

$$\bar{R}_{q,\hat{q}}(\delta) \geq R(\delta) \tag{6}$$

□

In the next step, we will prove that $\bar{R}_{q,\hat{q}}$ has the same properties as R for this new model.

Theorem 2. *$\bar{R}_{q,\hat{q}}(\delta)$ is the minimum number of bits required to represent a source sequence to keep the average distortion below δ.*

Proof. The proof is similar to the classic information theory case. Assume a compression scheme X that will compress k sequences into n bits, which can satisfy the distortion requirement. We know that the k sequences will become k transitory codewords. Note that $I(T;\hat{T}) \geq k\bar{R}_{q,\hat{q}}(\delta)$. Finally, $I(T;\hat{T}) \leq I(T;X) \leq H(X) \leq n$. Combining all the results, we get that $n/k \leq \bar{R}_{q,\hat{q}}(\delta)$. □

Before proving the source coding theorem for our model, we define the transitory distortion function as follows.

Definition (transitory distortion function)**.** *For the system of Figure 3, and distortion function d, the transitory distortion function is defined as*

$$d'(t,\hat{t}) = \sum_{s,s'\in\mathcal{S}} p(s,\hat{s}|t,\hat{t})d(s,\hat{s}) \tag{7}$$

Now, we prove a couple of important properties about this function:

Lemma 2. *(a)* $d'(t,\hat{t}) = \sum_{s,s'\in\mathcal{S}} \frac{p(s)q(t|s)}{\sum_{l\in\mathcal{S}} p(l)q(t|l)}\hat{q}(\hat{s}|\hat{t})d(s,\hat{s})$

(b) $E[d'(\mathbf{T},\hat{\mathbf{T}})] = E[d(\mathbf{S},\hat{\mathbf{S}})]$

(c) Let $\bar{\delta}_{min} = \min_{\hat{T}} E[d'(T,\hat{T})]$*. We have:*

$$\begin{aligned}\bar{\delta}_{min} &= \sum_{t\in\mathcal{T}} p(t)\min_{\hat{t}} d'(t,\hat{t}) \\ &= \sum_{t\in\mathcal{T}} \min_{\hat{t}}(\sum_{s,\hat{s}} \hat{q}(\hat{t}|\hat{s})p(s)q(t|s)d(s,\hat{s}))\end{aligned} \tag{8}$$

Proof. (a) First note that given t and $\hat{t}$, s and $\hat{s}$ are independent. Thus, $p(s,\hat{s}|t,\hat{t}) = p(s|t,\hat{t})p(\hat{s}|t,\hat{t}) = p(s|t)p(\hat{s}|\hat{t})$. Now, the second term in the equation is equal to $\hat{q}(\hat{s}|\hat{t})$ (by definition of the prebuilt decoder). The first term can be calculated by using the Bayes rule and the law of total probability (for the denominator).

(b) We have:

$$\begin{aligned}E[d'(\mathbf{T},\hat{\mathbf{T}})] &= \sum_{t,\hat{t}\in\mathcal{T}} p(t,\hat{t})d'(t,\hat{t}) \\ &= \sum_{t,\hat{t}\in\mathcal{T}} p(t,\hat{t}) \sum_{s,s'\in\mathcal{S}} p(s,\hat{s}|t,\hat{t})d(s,\hat{s}) \\ &= \sum_{t,\hat{t}\in\mathcal{T},s,s'\in\mathcal{S}} p(s,\hat{s},t,\hat{t})d(s,\hat{s}) \\ &= \sum_{s,s'\in\mathcal{S}} p(s,\hat{s})d(s,\hat{s}) \\ &= E[d(\mathbf{S},\hat{\mathbf{S}})]\end{aligned}$$

(c) Note that $E[d'(\mathbf{T},\hat{\mathbf{T}})] = \sum_{t,\hat{t}\in\mathcal{T}} p(t,\hat{t})d'(t,\hat{t}) \geq \sum_{t,\hat{t}\in\mathcal{T}} p(t,\hat{t})\min_u d'(t,u)$. Thus, it is easy to see that by choosing the test channel that maps t to $argmin_u d'(t,u)$ with probability 1, we will achieve this lower bound. The second line is reached simply by plugging in the definition for d'.

□

This lemma shows that the transitory distortion function is well defined (can be calculated) and has the same expected value as the original distortion function. Now we are ready to prove the source coding theorem for this model.

Theorem 3. *For a system of Figure 3, fix a distortion $\delta \geq \bar{\delta}_{min}$. Then for any $\delta' > \delta$ and $R' > \bar{R}_{q,\hat{q}}(\delta)$, for sufficiently large k there exists a source code C of length k with M codewords where,*

1) $M \leq 2^{\lfloor kR' \rfloor}$

2) $d(C) < \delta'$

Proof. First, imagine a source that produces sequences according to $\mathbf{T}$ and a distortion rate function d'. Call the rate distortion function for this source $R_T(\delta)$. Now, note that as $E[d(\mathbf{S},\hat{\mathbf{S}})] = E[d'(\mathbf{T},\hat{\mathbf{T}})]$, we will have $R_T(\delta) = \bar{R}_{q,\bar{q}}(\delta)$.

According to theorem 1, for any $\delta' > \delta$ and $R' > \bar{R}_{q,\bar{q}}(\delta)$, for sufficiently large k there exists a source code C of length k with M codewords where (1) $M \leq 2^{\lfloor kR' \rfloor}$, and (2) $d'(C) < \delta'$.

We argue that C satisfies the conditions for the system in Figure 3. Condition 1 is satisfied by default. The second condition follows from the fact that $d(C) = d'(C)$. □

Remark. *One can calculate the transitory rate distortion function using the Blahut-Arimoto Algorithm by using $p(T)$ instead of $p(S)$, and $d'(t,\hat{t})$ instead of $d(t,\hat{t})$.*

C. Restricted Rate Distortion Function

Here, we will introduce another variant of rate rate-distortion function that can be of mathematical and conceptual interest. Consider the system in Figure 3 again. Now, assume we are interested in seeing what the minimal mutual information possible between source and destination is while keeping the expected distortion below a certain level. To fully define this function, we first define the set of reachable destination distributions.

Definition. *We call a test channel $\hat{S}$ **reachable** from the source random variable S if there exists a test channel that will result in the destination being $\hat{S}$. More accurately, if $Q \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{T}|}$ is the transition matrix of the encoder and $\hat{Q} \in \mathbb{R}^{|\mathcal{T}|\times|\mathcal{S}|}$, the transition matrix of the decoder, $\hat{S}$ is reachable from S if and only if*

$$\exists C \in \mathbb{R}^{|\mathcal{T}|\times|\mathcal{T}|} : p(\hat{S}|S) = QC\hat{Q}$$

We call the set of reachable random variables $\mathcal{G}_{Q,\hat{Q}}(S)$.

Remark. *We can equivalently define the set of reachable test channels as the set of matrices H, where:*

$$\exists C \in \mathbb{R}^{|\mathcal{T}|\times|\mathcal{T}|} : H = QC\hat{Q}$$

We call this set $\mathcal{H}_{Q,\hat{Q}}$.

Now, we are ready to present the new function.

Definition (Restricted Rate Distortion Function)**.** *For a source S and prebuilt encoder Q and prebuilt decoder $\hat{Q}$, the restricted rate distortion function is defined as:*

$$\tilde{R}_{Q,\hat{Q}}(\delta) = \min_{\hat{\mathbf{S}}}\{I(\mathbf{S};\hat{\mathbf{S}}) : \hat{S} \in \mathcal{G}_{Q,\hat{Q}}(S), E[d(\mathbf{S},\hat{\mathbf{S}})] \leq \delta\} \tag{9}$$

Lemma 3. $\forall Q, \hat{Q}, \delta : R(\delta) \leq \tilde{R}_{Q,\hat{Q}}(\delta) \leq \bar{R}_{q,\hat{q}}(\delta)$

Proof. To prove the left inequality, note that the set of random variables available for $\tilde{R}_{Q,\hat{Q}}(\delta)$ is a subset of the available random variables for $R(\delta)$. The right inequality can be proven from the fact that as $\mathbf{S} \rightarrow \mathbf{T} \rightarrow \hat{\mathbf{T}} \rightarrow \hat{\mathbf{S}}$ is a Markov chain. Hence, for all test channels $I(T;\hat{T}) \geq I(S;\hat{S})$. ☐

Remark. *Note that $I(S;\hat{S})$ is also upper bounded by $I(S;T)$ and $I(\hat{S};\hat{T})$ (as can be seen in Figure 3). Thus, the bottleneck of communication can be the encoder or decoder, depending on the design of the system (other than the usual test channel).*

Remark. *The restricted rate distortion function can be viewed as a measure of compatibility between the prebuilt encoder and decoder. In other words, lower compatibility means that in order to preserve a certain average distortion, they have to communicate more bits(higher restricted rate distortion function).*

IV. CONNECTION TO SEMANTIC COMMUNICATION

In this section, we will explore the potential application of our model for semantic communication. We will draw our examples from human-to-human communication, but these arguments can be easily extended to human-to-machine and machine-to-machine communications.

The core of semantic communication is ”meaning”. In contrast to Shannon’s information theory, where the goal of communication is to accurately transmit symbols, semantic communication deals with the accuracy of communication in conveying the intended meaning.

In trying to convey meaning, the success of the communication and the amount of communication needed depend on multiple factors that cannot be changed during the communication. These things can be the world model, prior knowledge base, or the inference procedure of the communicating parties [9]. One can view them as the prebuilt encoder and decoder defined in this paper.

We give two examples to demonstrate this on both the source and destination sides.

Example 1: Assume the speaker wants to describe a ”cat”. However, because of their blurry vision, they classify this animal as a ”dog” in their mind. This is a failure in the inference procedure of the source. In our model’s terms, the prebuilt encoder encodes the ”cat” in the real world as a ”dog” in the speaker’s mind. Now, the communication of these parties should consider their blurry vision, and use broader words to account for the inaccuracy of inference (for example, use the more generic word ”animal” to describe what they see).

Example 2: Imagine a scenario in which the speaker uses the word ”tree” to describe a tropical tree (for example, a ”mango” tree). The listener, being from a colder climate, will translate the concept of a ”tree” into an evergreen tree, such as a ”pine” tree. This mismatch happens as a result of the two parties having different world models and different prior knowledge bases.

The concept and definition of ”semantic meaning” have been the topic of philosophical debate. One rather straightforward way to look at ”semantic meaning”, which fits well within our model, is to view it as external objects (or, equivalently, objective truths). In this view, the perception block of this communication will be the agent to analyze (and subsequently compress) these objects into mental images of them.

Our proposed model for semantic communication is summarized in Figure 4. The speaker will use their perception to encode the semantic meaning into a mental image/concept. This encoding will be affected by their inference process, etc., and is unchangeable for one round of communication. Now, the speaker chooses syntactic sentences to explain this mental image using their articulation agent. This sentence is then transmitted through a channel, which is the means of communication. The listener then decodes the noisy sentence received using their comprehension agent into a mental image. Finally, they project this translated concept/image into a semantic meaning.

Remark. *We can also define a distortion metric on these semantic meanings rather intuitively. This distortion function will serve as a closeness metric between these meanings. For instance, $d(\text{"cat"},\text{"dog"}) < d(\text{"cat"},\text{"tree"})$.*

Remark. *Analogous to the final remark in section III, one can view the restricted rate distortion function in this model, as a measure of compatibility between people. In other words, if the perception and projection blocks of the communicating parties are more compatible, they can communicate their intended meanings with less words.*

Remark. *This model can also be potentially used to describe how the meaning and interpretations of words change over time. One can imagine having a feedback loop in semantic communication (which is usually the case for this type of communication), and updates happening in the articulation and comprehension blocks based on this feedback.*

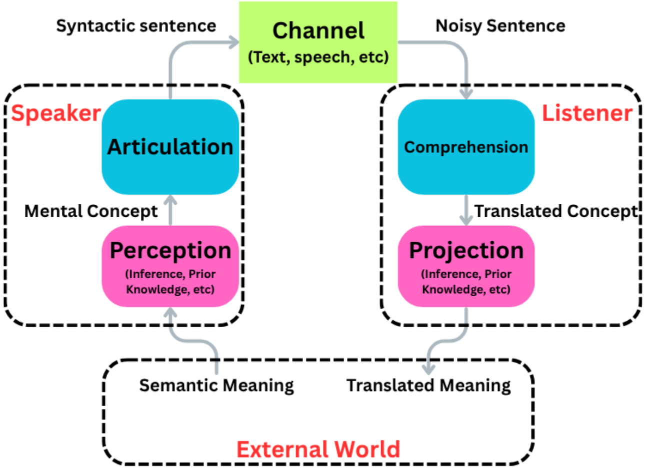

Fig. 4. Semantic communication model

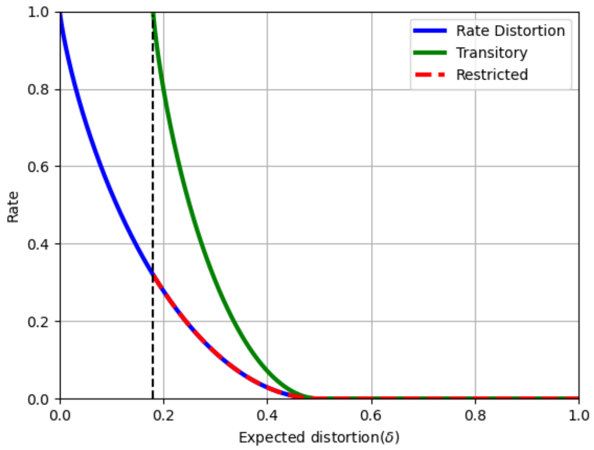

Fig. 5. Different distortion functions for the binary symmetrical source

V. NUMERICAL RESULTS

In this section, we will provide some numerical results to shed more light on our model.

A. Binary Symmetrical Source

Imagine a source $\mathcal{S} = \{0, 1\}$, with statistics $p(0) = p(1) = 0.5$ and $d(0, 1) = 1$. Also, let the prebuilt encoder and decoder be binary symmetric channels with a probability of error equal to 0.1. We will calculate the rate distortion, transitory rate distortion, and restricted rate distortion function for this model.

It is a well-known result that the rate distortion for this source is:

$$R(\delta) = \begin{cases} 1 - H(\delta), & 0 \leq \delta \leq 0.5 \\ 0, & \delta > 0.5 \end{cases} \tag{10}$$

And the optimal test channel is a binary symmetric channel with probability of error equal to δ.

Now to calculate the transitory rate distortion function, notice that the transitory codewords are $\mathcal{T} = \{0, 1\}$ with statistics $p(0) = p(1) = 0.5$. The transitory distortion function is:

$$d'(t, t') = \begin{cases} 0.18, & t = t' \\ 0.82, & t \neq t' \end{cases} \tag{11}$$

Now, given this information, it is easy to see that the transitory rate distortion function will be:

$$\bar{R}_{q,\hat{q}}(\delta) = \begin{cases} 1 - H(\frac{\delta - 0.18}{0.64}), & 0.18 \leq \delta \leq 0.5 \\ 0, & \delta > 0.5 \end{cases} \tag{12}$$

To find the restricted rate distortion function, we first find the set of reachable test channels. We show a channel with $p(1|0) = a$ and $(0|1) = b$ as $C_{a,b}$. It is easy to verify that the reachable set is:

$$H_{Q,\hat{Q}} = \{C_{a,b} : 1.8 \leq 9a + b \leq 8.2, 1.8 \leq 9b + a \leq 8.2\} \tag{13}$$

Thus, it is easy to show that (1) For $\delta \geq 0.18$, the optimal test channel is reachable. (2) For $\delta < 0.18$, no test channel is reachable that achieves this expected distortion. Thus:

$$\tilde{R}_{Q,\hat{Q}}(\delta) = \begin{cases} R(\delta), & \delta \geq 0.18 \\ undefined, & OW \end{cases} \tag{14}$$

These functions are illustrated in Figure 5, and the set of reachable test channels can be seen in Figure 6 (a channel (a,b) is reachable if it's shaded blue).

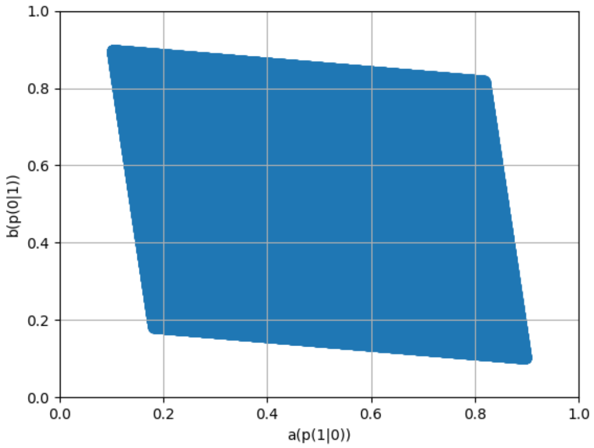

Fig. 6. The region of reachable test channels

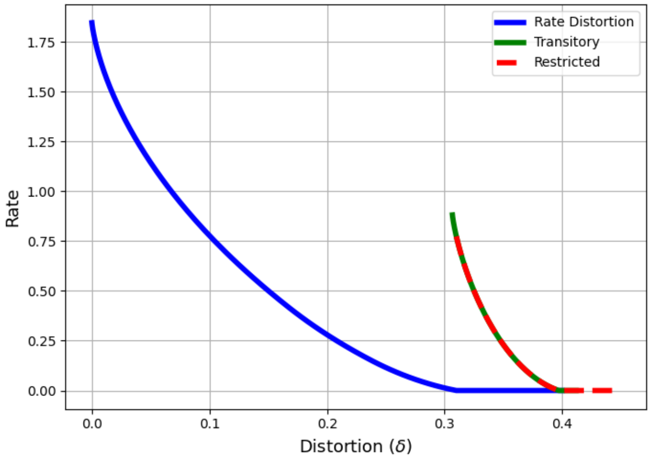

Fig. 7. Distortion functions for the suboptimal compressor

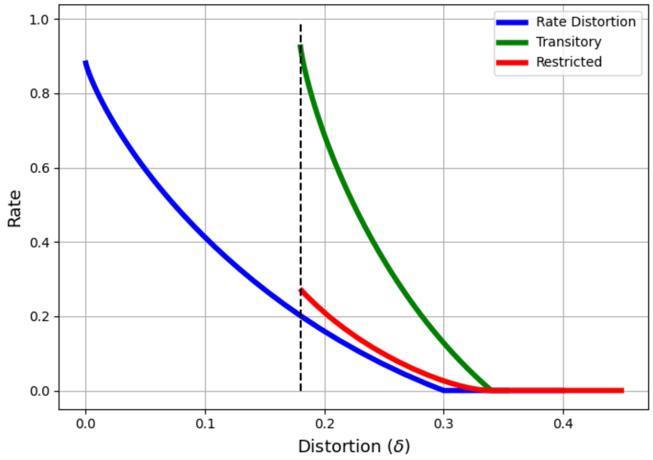

Fig. 8. Different distortion functions for the binary unsymmetrical source

B. Suboptimal Compressor

To see another example, imagine a source $\mathcal{S} = \{1, 2, 3, 4\}$ where $p(k) = 0.1 * k$. The distortion matrix D is given below. Also, let the prebuilt encoder be a compressor that encodes 1 and 2 into the transitory codeword A and 3 and 4 into B. The decoder will decode the transitory codeword according to the Bayes rule. In other words $\hat{q}(s|t) = \frac{p(s)q(t|s)}{p(t)}$.

$$D = \begin{bmatrix} 0 & 0.5 & 0.6 & 1 \\ 0.5 & 0 & 0.2 & 0.5 \\ 0.6 & 0.2 & 0 & 0.7 \\ 1 & 0.5 & 0.7 & 0 \end{bmatrix}$$

The distortion functions can be seen in Figure 8. As it can be seen, in this example $\bar{R} = \tilde{R}$, as opposed to the previous example where $\tilde{R} = R$. In the next example, we will provide a case where the restricted rate distortion function is strictly between the normal rate distortion and the transitory one.

C. Binary unsymmetrical source

To see a case where $R(\delta) < \tilde{R}_{Q,\hat{Q}}(\delta) < \bar{R}_{q,\hat{q}}(\delta)$, imagine the same setup as in V-A, but with $p(0) = 1 - p(1) = 0.7$. The set of reachable test channels is the same as before (Figure 6) but the rate distortion functions are different and can be seen in Figure 8. Thus, throughout these numerical examples, we have shown that the restricted distortion function can take all three forms.

VI. CONCLUSION

In this paper, we looked into communication systems with prebuilt components (specifically prebuilt encoders and decoders). First, we proposed a novel model to capture this concept. The cost and reward of this communication were defined. Using these, we introduced the optimization problem for this model.

Next, we looked into the case with the perfect channel. We defined two novel rate-distortion functions: (1) The transitory rate-distortion function, which was proved to be the fundamental limit of compressing ”transitory codewords”. (2) The restricted rate-distortion function, which is the minimum rate of compression available for source sequences given the prebuilt encoder and decoder. We proved that the restricted rate distortion function is upper bounded by the transitory one. Furthermore, we showed that this restricted rate is lower bounded by the classical rate distortion function (which is intuitive as the prebuilt components may be suboptimal).

Finally, we explored the connection of our model to semantic communication. We argued that the heart of semantic communication is the components that are not changeable. These components are affected by the inference process, world model, and prior knowledge of the communicating parties. The existence of these components will motivate using our model instead of the classic model.

One line of future work is modeling the existing components of communication systems more rigorously. These components can take complex form such as LLMs, generative models, or Neural Networks. Understanding the structure and limitations of this components is crucial.

REFERENCES

[1] R. McEliece, *The source–channel coding theorem*, p. 112–122. Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2002.

[2] H. Xie, Z. Qin, G. Y. Li, and B.-H. Juang, “Deep learning enabled semantic communication systems,” *IEEE Transactions on Signal Processing*, vol. 69, pp. 2663–2675, 2021.

[3] Z. Weng and Z. Qin, “Semantic communication systems for speech transmission,” *IEEE Journal on Selected Areas in Communications*, vol. 39, no. 8, pp. 2434–2444, 2021.

[4] K. Lu, Q. Zhou, R. Li, Z. Zhao, X. Chen, J. Wu, and H. Zhang, "Rethinking modern communication from semantic coding to semantic communication," *IEEE Wireless Communications*, vol. 30, no. 1, pp. 158–164, 2023.

[5] F. Lotfi, O. Semiari, and W. Saad, "Semantic-aware collaborative deep reinforcement learning over wireless cellular networks," *ICC 2022 - IEEE International Conference on Communications*, pp. 5256–5261, 2021.

[6] R. Carnap and Y. Bar-Hillel, "An outline of a theory of semantic information," *Journal of Symbolic Logic*, vol. 19, no. 3, pp. 230–232, 1952.

[7] L. Floridi, "Outline of a theory of strongly semantic information," *Minds and Machines*, vol. 14, no. 2, pp. 197–221, 2004.

[8] S. D'Alfonso, "On quantifying semantic information," *Information*, vol. 2, no. 1, pp. 61–101, 2011.

[9] J. Bao, P. Basu, M. Dean, C. Partridge, A. Swami, W. Leland, and J. A. Hendler, "Towards a theory of semantic communication," in *2011 IEEE Network Science Workshop*, pp. 110–117, 2011.

[10] B. Güler, A. Yener, and A. Swami, "The semantic communication game," *IEEE Transactions on Cognitive Communications and Networking*, vol. 4, no. 4, pp. 787–802, 2018.

[11] K. Niu and P. Zhang, "A mathematical theory of semantic communication," 2024.

[12] Y. Shao, Q. Cao, and D. Gunduz, " A Theory of Semantic Communication ," *IEEE Transactions on Mobile Computing*, vol. 23, pp. 12211–12228, Dec. 2024.

[13] R. McEliece, *Discrete memoryless sources and their rate-distortion functions*, p. 75–94. Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2002.