A Noncooperative Solution for Coalition Formation and Payoff Negotiation

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***Abstract*—We propose a noncooperative solution concept for cooperative games. By treating both the partitioning of players into coalitions and the intra-coalition payoff allocations as state variables, our approach captures their intrinsic interdependence and determines their equilibrium outcomes simultaneously. Cast as a dynamic game, the model naturally incorporates players' farsighted behavior. Because the system's state reflects the entire player partition, the framework readily accommodates coalitional games with externalities.**

***Index Terms*—noncooperative dynamic games, farsightedness, externality**

I. Introduction

Cooperative game theory offers a range of solution concepts that can be broadly divided into two categories: those that aim to predict likely outcomes—such as the core and the stable set—and those that prescribe normative outcomes—such as the Shapley value. The solution proposed in this paper falls into the former category.

Our objective is to establish a noncooperative foundation for cooperative games. Whereas standard coalitional analysis assumes that binding contracts can be written and enforced within a coalition, it has long been recognized that sustained interaction and competition can foster cooperation based purely on individual self-interest. It is therefore both natural and feasible to seek a noncooperative basis for cooperative games. The core exemplifies this approach by identifying equilibrium allocations from which no group of players can deviate to obtain a better outcome, albeit under the restrictive assumption that the grand coalition has already formed.

Players' payoffs are inherently tied to the coalitions they form. Given a coalition structure, payoffs represent the allocation of the values generated by different coalitions. Conversely, if each player is rational and seeks to maximize their own payoff, coalition formation itself depends on how these values are distributed among individual players. Our model captures this interdependence by treating both payoff allocations and player partitions as state variables, allowing their equilibrium values to be determined *jointly*.

For illustration, consider transferable-utility (TU) coalitional games. The outcome of such a game has two components: the coalitions that form and the payoffs each player receives. Classical solution concepts such as the core and the stable set describe outcomes solely through the payoff vector and are therefore insufficient to capture the full picture. By contrast, our framework is readily applicable to coalitional games with externalities, because its state variable encompasses the entire player partition. This allows it to determine each coalition's value as a function not only of its own composition but also of its "environment", i.e., the other coalitions that form.

The proposed solution is farsighted. Whereas existing concepts such as the core and the stable set rely on "myopic" preferences, truly rational players should be patient and forward-looking, focusing on their long-term payoffs. Incorporating farsightedness also alleviates the existence problems that plague many equilibrium concepts. For instance, what does it mean for the core to be empty—and what, then, is the outcome of the game? By using both the player partition and the payoff vector as state descriptors, our model fully captures the dynamic process of "dominance," in which players prefer one state over another. The steady state of this process constitutes the farsighted outcome of the game.

Two main approaches have been used to study cooperative games: one based on noncooperative bargaining and the other on blocking coalitions. The bargaining approach often requires specifying the bargaining process, whereas the blocking approach emphasizes coalitional dominance and is therefore closer to our perspective. Several blocking-based models incorporate farsightedness [1]–[4], and coalition formation as a dynamic process is examined in [5]. Our work differs from these studies in that we adopt a noncooperative view of the problem rather than treating coalitions as basic behavioral units. This stance allows us to sidestep several challenges inherent in the blocking approach, such as issues of consistency and maximality [3], [4]. Conceptually, our framework blends elements of noncooperative bargaining and blocking coalitions into what might be called "noncooperative blocking," where bargaining outcomes are constrained solely by the payoffs players would receive if negotiations were to fail. In this sense, our proposed solution generalizes the feasible set of the two-person Nash bargaining problem to games with an arbitrary number of players. Unlike the two-person case, where the only options are to cooperate or separate, the withdrawal of some players in a multi-player setting can trigger cascading reactions from others, rendering the feasible set far less transparent.

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II. Model

A. Notation

We focus on a coalitional game with transferable payoffs. Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. Any subset of N is called a coalition, and the set N itself is the grand coalition.

A characteristic-function game is a pair (N, v), where $v : 2^N \to \mathbb{R}$ assigns a real number to each coalition, with the convention that $v(\varnothing) = 0$. The function v is called the characteristic function.

An outcome of the game consists of two components: the coalitions formed by the players and the payoff allocated to each player. The first component is represented by a partition $\pi = \{S_1, \ldots, S_k\}$ of the player set N, where $S_i \cap S_j = \varnothing$ for all $i \neq j$ and $\cup_{i=1}^{k} S_i = N$. In other words, each player belongs to exactly one coalition. The set of all possible partitions of a set S is denoted by $\Pi(S)$. We use $\pi_1 \in \Pi(S)$ to denote the partition in which every element of π_1 is a singleton coalition.

The second component, the payoff allocation, can be represented by a vector $x \in \mathbb{R}^n$. For any subset $S \subseteq N$, let $x(S) = \sum_{i \in S} x_i$. A pair $(x, \pi) \in \mathbb{R}^n \times \Pi(N)$ is a feasible outcome of the game if

$$x(S) = v(S), \quad \forall S \in \pi.$$

Thus, each coalition's allocation must be efficient, with no transfer of payoff across different coalitions.

A state of the game is described by a partition $\pi \in \Pi(N)$. For each state, every player i has a set of available actions $A(i) = \{a_1(i), a_2(i), \ldots, a_{2^{n-1}}(i)\}$. [1] Here $a_l(i)$ represents the l-th coalition that includes player i, choosing $a_l(i)$ indicates player i's decision to join that coalition.

We define $\Delta(A(i))$, the set of probability distributions in $A(i)$, as the set of mixed strategies available to player i. A mixed strategy vector $\sigma(i, k)$ for player i in state k, assigns probability $\sigma_l(i, k)$ to action $a_l(i) \in A(i)$. We sometimes write the state π explicitly instead of its index k, and the coalition $a_l(i) = S$ instead of its index l. Formally, we have

$$\Delta(A(i)) = \{\sigma(i,k) \in \mathbb{R}^{2^{n-1}} : \sigma_l(i,k) \geq 0 \quad \text{for all } l = 1, 2, \ldots, 2^{n-1} \text{and} \sum_{l=1}^{2^{n-1}} \sigma_l(i,k) = 1\}$$

B. Assumptions

Although our formulation applies to general dynamic games, for clarity of exposition we restrict attention to stochastic games in which history-dependent strategies are ruled out. At each stage of the game, players have complete information and act simultaneously.

Each player cares only about their long-term payoff, defined as the average reward received over an infinite horizon. The model and its solution extend naturally to finite-horizon settings and to games with discounted cumulative payoffs.

Given the strategy profile of all players, the dynamics of the stochastic game are fully characterized by the transition probabilities of a Markov chain whose states correspond to the possible partitions of the game. For a given k-th state π and each player i's mixed strategy $\sigma_l(i, k)$, if each player i chooses its l_i-th action, the system transitions to (possibly) a different state π'. The specification of π' is a modeling question. We adopt the assumption that a non-singleton coalition $S \in \pi'$ forms if and only if all members simultaneously choose to form that coalition. Formally, for any $S \in N$ with $|S| \geq 2$, we have $S \in \pi'$ if and only if, for any player $i \in S$, the action $a_{l_i}(i, k)$ corresponds to joining coalition S. This rule defines a mapping $f(l_1, l_2, \ldots, l_n)$ from any action profile $(l_1, l_2, \ldots, l_n)$ to a unique partition π'. [2]

Define the indicator function

$$\mathbb{1}(\pi | l_1, l_2, \ldots l_n) = \begin{cases} 1, & \text{if } \pi = f(l_1, l_2, \ldots, l_n) \\ 0, & \text{otherwise} \end{cases}.$$

We then have the transition probability from state π to state π' under the mixed strategies $\{\sigma(i, \pi)\}_{i \in N}$ is

$$P(\pi'|\pi) = \begin{cases} \displaystyle\sum_{l_1, l_2, \ldots, l_n} \mathbb{1}(\pi' | l_1, l_2, \ldots, l_n) \prod_{i=1}^{n} \sigma_{l_i}(i, \pi), & \pi' \neq \pi_1 \\ 1 - \displaystyle\sum_{\pi'' \neq \pi_1} P(\pi''|\pi), & \pi' = \pi_1 \end{cases}.$$

C. Equilibrium

With the transition probabilities $P(\pi'|\pi)$ for all states π and π', the steady-state distribution of the Markov chain, denoted by the row vector μ, is obtained from

$$\mu P = \mu.$$

The long-term (average) payoff of player i is then

$$U_i = \sum_{\pi} \mu(\pi) x_i(\pi),$$

where $x_i(\pi)$ is the payoff to play i in state π.

Because μ depends on the players' mixed strategies, U_i can be expressed more explicitly as $U_i(x, \sigma)$, highlighting that player i's long-term payoff is a function of the allocation vector $x = \{x(\pi)\}_{\pi}$ and the strategy profile $\sigma = \{\sigma(i, \pi)\}_{i,\pi}$.

The equilibrium outcome of the game is a feasible payoff allocation $x = \{x(\pi)\}_{\pi}$ together with a strategy profile $\sigma = \{\sigma(i, \pi)\}_{i,\pi}$ such that no subgroup of players can profitably deviate. Formally, (x, σ) is an equilibrium if there is no coalition $S \in N$ and alternative feasible payoff allocation x' and strategy profile σ' satisfying

$$U_i(x', \sigma') > U_i(x, \sigma) \quad \forall i \in S$$

with $x'_i(\pi) = x_i(\pi)$ and $\sigma'(i, \pi) = \sigma(i, \pi)$ for all $i \notin S$ and all $\pi \in \Pi(N)$.

[1] In general, the set of available actions $A(i, \pi)$ depends on the state π. In this paper, however, we focus on the case in which $A(i, \pi)$ is the same for all states, and thus we simply write $A(i)$.

[2] In general, this mapping may depend on the current state π. In this paper, however, we assume that the players' action profile alone fully determines the next state. So f does not depend on π.

III. EXAMPLES

A. Example 1 (n=2)

We first solve our model for the smallest nontrivial case, $N = \{1, 2\}$. The set of all possible partitions of N is $\Pi(N) = \{\pi_1 = \{\{1\}, \{2\}\}, \pi_2 = \{\{1, 2\}\}\}$. The sets of available actions are $A(1) = \{\{1\}, \{1, 2\}\}$ and $A(2) = \{\{2\}, \{1, 2\}\}$

For notational simplicity, define $\sigma_2(1, 1) = \alpha_1$ and $\sigma_2(1, 2) = \beta_1$, then $\sigma_1(1, 1) = 1 - \alpha_1$ and $\sigma_1(1, 2) = 1 - \beta_1$. Similarly, we have $\sigma_2(2, 1) = \alpha_2$ and $\sigma_2(2, 2) = \beta_2$, then $\sigma_1(2, 1) = 1 - \alpha_2$ and $\sigma_1(2, 2) = 1 - \beta_2$. The corresponding state-transition matrix is

$$P = \begin{bmatrix} 1 - \alpha_1\alpha_2 & \alpha_1\alpha_2 \\ 1 - \beta_1\beta_2 & \beta_1\beta_2 \end{bmatrix}.$$

The steady-state probabilities of the system are

$$\mu_1 = \frac{1 - \beta_1\beta_2}{1 + \alpha_1\alpha_2 - \beta_1\beta_2}, \quad \mu_2 = \frac{\alpha_1\alpha_2}{1 + \alpha_1\alpha_2 - \beta_1\beta_2}.$$

The expected payoffs of the two players are then

$$U_i = \mu_1 v(\{i\}) + \mu_2\gamma_i v(\{1, 2\}), \quad i = 1, 2$$

where γ_1 and γ_2 denote the shares of $v(\{1, 2\})$ allocated to players 1 and 2, respectively, with $\gamma_1 + \gamma_2 = 1$.

There are three possible scenarios:

- If $v(\{1\}) > v(\{1, 2\})$ or $v(\{2\}) > v(\{1, 2\})$, i.e., at least one player's stand-alone value exceeds the value produced by the grand coalition. Without loss of generality assume $v(\{1\}) > v(\{1, 2\})$. Then the optimal $\alpha_1 = 0$, $\beta_1 < 1$. The outcome of the game is $(\pi, x) = (\{\{1\}, \{2\}\}, (v\{1\}, v\{2\}))$.
- If $\max(v(\{1\}), v(\{2\})) \leq v(\{1, 2\}) < v(\{1\}) + v(\{2\})$, then at least one player still obtains more value by remaining alone rather than joining the grand coalition. Again the outcome is $(\pi, x) = (\{\{1\}, \{2\}\}, (v\{1\}, v\{2\}))$.
- If $v(\{1, 2\}) \geq v(\{1\}) + v(\{2\})$, there exist γ_1 and γ_2 such that $\gamma_1 v(\{1, 2\}) > v(\{1\})$ and $\gamma_2 v(\{1, 2\}) > v(\{2\})$. Then the optimal $\alpha_1 < 1$, $\beta_1 = 1$, $\alpha_2 < 1$, $\beta_2 = 1$. The outcome of the game is $(\pi, x) = (\{1, 2\}, (\gamma_1 v(\{1, 2\}), \gamma_2 v(\{1, 2\})))$. In this third case, the set of possible payoff vectors coincides exactly with the core of the game. Note that when $v(\{1, 2\}) = v(\{1\}) + v(\{2\})$, the outcome $(\pi, x) = (\{\{1\}, \{2\}\}, (v\{1\}, v\{2\}))$ is also a possible equilibrium.

B. Example 2 (n=3, with externalities) [6]

For $N = \{1, 2, 3\}$, the set of all possible partitions of is $\Pi(N) = \{\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}, \{\{1, 2, 3\}\}\}$, which we label $\pi_1, \ldots, \pi_5$ respectively. The characteristic function specifies $v(\{1, 2\}) = 12$, $v(\{1, 3\}) = 13$, $v(\{2, 3\}) = 14$, and $v(\{1, 2, 3\}) = 24$. It is assumed that a single player can generate nothing by himself, but he can obtain a payoff of 9 by freeriding on the public good produced by the other two players if they form a coalition. This captures the externality imposed by the coalition on the outsider. To capture such externalities, the value assigned to a coalition must be specified relative to the entire partition of players. For example, $v(\{1\}, \{\{1\}, \{2, 3\}\}) = v(\{1\}, \pi_4) = 9$, and similarly, $v(\{2\}, \pi_3) = v(\{3\}, \pi_2) = 9$.

In [6], the author argued that with arrival order 1, 2, 3, two separate coalitions will form: $\{2; 3\}$ and $\{1\}$, yielding payoffs of 9, 10, and 4 to players 1, 2, and 3, respectively. Player 3 accepts 4 because the model does not consider farsightedness. Comparable results arise for the other five arrival orders.

We now apply our solution to this example. To simplify the analysis, we assume that players' strategies are state independent, i.e., $\sigma_l(i, k) = \sigma(i)$. Under this assumption, the Markov game reduces to a repeated game. It then follows that $\mu_2 = \sigma_{12}(1)\sigma_{12}(2)$, $\mu_3 = \sigma_{13}(1)\sigma_{13}(3)$, $\mu_4 = \sigma_{23}(2)\sigma_{23}(3)$, $\mu_5 = \sigma_{123}(1)\sigma_{123}(2)\sigma_{123}(3)$, and $\mu_1 = 1 - \sum_{i=2}^{5} \mu_i$.

Is $((9, 10, 4), \pi_4)$, obtained when $\sigma_{23}(2) = \sigma_{23}(3) = 1$, an equilibrium? It is not. Consider $S = N$, and set $\sigma_{123}(1) = \sigma_{123}(2) = \sigma_{123}(3) = 1$, $x_1(\pi_5) > 9$, $x_2(\pi_5) > 10$, and $x_3(\pi_5) > 4$, which is feasible given $x_1(\pi_5) + x_2(\pi_5) + x_3(\pi_5) = 24$, All players are better off. This suggests that under farsighted behavior the eventual equilibrium payoffs are likely to be more balanced than (9,10,4).

IV. CONCLUSION

We introduce a noncooperative solution concept for joint coalition formation and payoff negotiation, providing a foundation for cooperative games. Given negotiated values in each state, the resulting outcome may be referred to as a Markov-perfect strong equilibrium, or a subgame-perfect strong equilibrium in general dynamic game settings. An interesting next step is to extend the framework to scenarios without characteristic functions, where a player's payoff depends on the actions that all the players take and on the cross-payments exchanged among them. In such environments, the very notion of a "coalition" may cease to apply, as a player could "cooperate" with multiple groups simultaneously, depending on their actions and how those actions are interpreted.

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