Recursive Parameter Identification Algorithm Stability Analysis Via Pi-Sharing

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Abstract—**Pi-sharing is introduced as an extension of the application of passivity and hyperstability concepts for discrete-time systems, providing connections between input-output and state-space stability notions. Using tools developed within the pi-sharing framework, new stability results for the output error class identification algorithm are derived. This approach offers a clear interpretation of the role of the SPR condition in the work of Landau and Silveira and its absence in the work of Tomizuka and Altay.**

I. Introduction

"You don't know how to manage Looking-glass cakes. Hand it round first, and cut it afterwards."—Unicorn to Alice in L. Carroll's *Through the Looking Glass.*

THE properties of passive systems and their interconnections have been studied for many years, beginning with insights developed in network theory and later generalized to more formal system theoretic terms, e.g., [1], [2], [10], [16]. Similarly, the related concept of hyperstability, since its introduction by Popov [3], has been employed in a wide range of problems, being particularly useful in treating systems containing certain types of nonlinearities. Landau and co-workers have successfully applied hyperstability theory to obtain stability conditions for an important class of parameter identification problems [4]. These results, although obtained by algebraic manipulations, closely resemble those obtained by passivity formulations. This link prompted formalization of the π-sharing concept described in this paper. The π-sharing approach relies on inequalities concerning a system input–output inner product having the physical interpretation of "energy" supplied to the system [1], [10], [11], [22]. This results in simple characterizations of system interconnections in terms of the "capacity to dissipate energy," which may be shared with deficiencies in this regard among connected subsystems. Like hyperstability, the behavior of the system state is associated with the energy supply inner product in π-sharing, relating input–output and state-space stability notions. However, π-sharing extends the application of both passivity and hyperstability for feedback connections, as we will show.

The π-sharing approach differs principally in viewpoint with the closely related dissipative systems approach [10], [13], [15]–[18]. The representation of the "energy supply" to a system is central to the distinction. While the dissipative systems approach has provided compact generalizations of many system characterizations and stability results, it is argued here that the π-sharing view more directly encourages an intuitive interpretation for the identification algorithms considered.

Section II of this paper provides a formal introduction to π-sharing, deriving tools for determining the π-sharing, hyperstability, passivity, and stability of interconnections of π-sharing systems. Also, similarities with the dissipative approach are discussed. Section III illustrates the use of the π-sharing approach by deriving results similar to [4], [5], together with some new results, for a class of recursive parameter identification algorithms. This application illuminates some essential differences between the π-sharing and hyperstability formulations. The conclusion raises some additional related applications, and indicates some directions for further work.

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II. Pi-Sharing Concepts and Results

We begin by discussing some notation and the class of systems under consideration. For clarity, only finite-dimensional, single-input single-output, discrete-time systems will be treated. Linearity and time-invariance, however, are not assumed. We will view the input u, output y, and the state trajectory x of a system S as functions from the natural numbers $\mathbb{N}$ into $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{R}^n$, respectively, for some finite state-space dimension n. It will be convenient to deal with the following collections of such functions [1].

Linear function (vector) space: $\mathbb{L}$, $\mathbb{L}_n$

$\mathbb{L}$ = {all functions $v: \mathbb{N} \to \mathbb{R}$, under pointwise addition and scalar ($\mathbb{R}$) multiplication}.

$\mathbb{L}_n$ = {all functions $v: \mathbb{N} \to \mathbb{R}^n$, under pointwise, componentwise addition, and scalar ($\mathbb{R}$) multiplication}.

Inner product space: $\mathbb{J}^2$, $\mathbb{J}_n^2$

$$\mathbb{J}^2 = \{v \in \mathbb{L} : \langle v, v\rangle < \infty\}; \; \mathbb{J}_n^2 = \{v \in \mathbb{L}_n : \langle v, v\rangle < \infty\}$$

where the inner product $\langle , \rangle$ is

$$\langle v, w\rangle \triangleq \sum_{k=0}^{\infty} v^T(k) w(k), \qquad \forall\, v, w \in \mathbb{L} \text{ or } \mathbb{L}_n. \tag{2-1}$$

The induced norm on $\mathbb{J}$ or $\mathbb{J}_n$ is given by

$$\|v\| \triangleq \langle v, v\rangle^{1/2}, \qquad v \in \mathbb{J} \text{ or } \mathbb{J}_n. \tag{2-2}$$

Similar inner product space structure on subspaces of $\mathbb{L}$ or $\mathbb{L}_n$ is provided by the truncated inner product

$$\langle v, w\rangle_T \triangleq \sum_{k=0}^{T} v^T(k) w(k), \qquad \forall v, w \in \mathbb{L} \text{ or } \mathbb{L}_n, \; \forall\, T \in \mathbb{N} \tag{2-3}$$

which will be employed throughout the discussion and results to follow.

We avoid issues of existence of solutions by making the following assumption.

Assumption: S can be described by

$$\left.\begin{aligned} x(k+1) &= f[x(k), u(k), k] \\ y(k) &= h[x(k), u(k), k] \end{aligned}\right\} \tag{2-4}$$

where $f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}^n$ and $h: \mathbb{R}^n \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$. Thus, for all $u \in \mathbb{L}$, and all $x(0) \in \mathbb{R}^n$ there exist unique $x \in \mathbb{L}_n$ and y

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$\in \mathbb{L}$ satisfying S. Also, we have the existence of truncated norms and inner products, e.g., $\|v\|_T \triangleq \langle v, v\rangle_T^{1/2}$ and $\langle v, w\rangle_T$, for all $T \in \mathbb{N}$ and all inputs, outputs, or states v, w concerning the system S. The results below can be extended to include relational operators S—a significant strength of the approach—but one which we defer in the interest of clarity. See, e.g., [22], [23] for a similar framework involving relations.

Consider two such systems S_1 and S_2 connected in the feedback configuration as in Fig. 1. We assume the composite system S satisfies the above assumption (2-4) with respect to the composite state $x = [x_1^T, x_2^T]^T$, input $u = u_1 + y_2$, and output $y = y_1 = u_2$. The π-sharing approach is characterized by the interpretation of the system input/output inner product (2-3) as the "energy supplied to the system up to time T." Considering a system as, say, an admittance function mapping voltages (inputs) to currents (outputs) at a port of an electrical network, this view is justified according to the physical definitions of power and energy. See [1], [11], [22] for additional discussion of this viewpoint. Formally, by the bilinearity and symmetry of the inner product $\langle\cdot, \cdot\rangle_T$ we have

$$\begin{aligned}\langle u, y\rangle_T &= \langle u_1+y_2, y\rangle_T = \langle u_1, y\rangle_T + \langle y_2, y\rangle_T \\ &= \langle u_1, y_1\rangle_T + \langle u_2, y_2\rangle_T \end{aligned} \tag{2-5}$$

which says the energy supplied to the composite system is simply the sum of the energy supplied to the subsystems. This linear energy supply relationship is the basis of the results in passivity theory for interconnections (e.g., [1], [10], [22] and generalizations [2], [13], [17]). We incorporate this basis into a framework involving the system state in the following definition of π-sharing. The "weighted norms" on $\mathbb{L}$, $\mathbb{L}_n$, and $\mathbb{R}^n$ employed below are written

$$(a)\|v\|_T^2 \triangleq \sum_{k=0}^{T} a(k)v^2(k), \qquad v \in \mathbb{L} \tag{2-6}$$

$$(A)\|v\|_T^2 \triangleq \sum_{k=0}^{T} v^T(k)A(k)v(k), \qquad v \in \mathbb{L}_n \tag{2-7}$$

$$(A)|v(k)|^2 \triangleq v^T(k)A(k)v(k), \qquad v(k) \in \mathbb{R}^n \tag{2-8}$$

where $a: \mathbb{N} \to \mathbb{R}$ and $A: \mathbb{N} \to \mathbb{R}^{n \times m}$ (symmetric) are any scalar and symmetric matrix sequences, respectively. When the weighting functions a and A become identity operators, we have the usual (induced) norms on these inner product spaces. This simplified notation is meant to retain the notion of modulation of the norms by the weighting functions. Let $C = \{(u, x, y) \in \mathbb{L} \times \mathbb{L}_n \times \mathbb{L}\}$ and $C_s = \{(u, x, y) \in C$ satisfying $S\}$.

Definition: Given $r: \mathbb{N} \to \mathbb{R}$, $p: \mathbb{N} \to \mathbb{R}$, and symmetric positive semidefinite (p.s.d.) $\Gamma: \mathbb{N} \to \mathbb{R}^{n \times n}$ and $Q: \mathbb{N} \to \mathbb{R}^{n \times n}$, a system S with input u, output y, and state x as assumed in (2-4) is said to be *π-sharing with respect to* (Γ, Q, p, r) iff

$$\begin{aligned}\langle u, y\rangle_T \geqslant{}& (\Gamma)|x(T+1)|^2 - (\Gamma)|x(0)|^2 \\ &+ (Q)\|x\|_T^2 + (p)\|y\|_T^2 + (r)\|u\|_T^2 \end{aligned} \tag{2-9}$$

holds $\forall T \in \mathbb{N}$ and all $(u, x, y) \in C_S$. The functions (Γ, Q, p, r) are called π-coefficients for S, and S will be simply termed π-sharing if the association to particular π-coefficients is clear.

It is evident from (2-5) that every feedback interconnection (Fig. 1) exchanges energy between subsystems. The π-coefficients p and r parameterize this exchange, while the Γ and Q coefficients parameterize the energy storage and dissipation properties of the system state trajectory. Note that for any system S, one could choose $Q(k) \equiv \Gamma(k) \equiv O$, $r(k) \equiv p(k) \equiv -1/2$ so

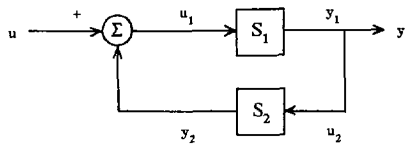

Fig. 1. Feedback connection.

that for all k

$$\begin{aligned}u(k)y(k) \geqslant{}& r(k)u^2(k) + p(k)y^2(k) + x^T(k)Q(k)x(k) \\ &+ x^T(k+1)\Gamma(k+1)x(k+1) - x^T(k)\Gamma(k)x(k) \\ ={}& -\frac{1}{2}u^2(k) - \frac{1}{2}y^2(k) \end{aligned} \tag{2-10}$$

and the inequality (2-9) holds for all $(u, x, y) \in C$. Thus, every system S under the assumption of (2-4) is π-sharing with respect to $(O, O, -1/2, -1/2)$, making this choice a rather useless indication of system energy exchange and storage properties. We shall discuss techniques for finding more descriptive sets of π-coefficients, making (2-9) a more useful system characterization.

Connections with Passivity, Hyperstability, Dissipativity, and Stability

The concepts of passivity, hyperstability, and dissipativity supply important interpretations of the π-coefficients characterizing a system. Positive π-coefficients indicate "energy dissipative" properties in a system, while negative coefficients allow energy to be generated. Remember that the "storage" coefficients Γ and Q are at least positive semidefinite by definition.

Lemma 1: S is passive [strictly passive] if there exists an $r_o \geqslant 0$ [$r_o > 0$] such that S is π-sharing with respect to (Γ, Q, p, r) where $r(k) \geqslant r_o$, $\forall k$, and $p(k) \geqslant 0$, $\forall k$.

Proof: See the Appendix. △△△

For a strictly passive system, then

$$\langle u, y\rangle_T \geqslant r_o\|u\|_T^2 + \beta, \qquad \forall\, T \in \mathbb{N}, \text{ some } r_o > 0,\ \beta \in \mathbb{R},$$

and the energy supplied to the system is greater than a function depending on the "size" of the input u and the constant β. With initial stored energy $-\beta$, the system may supply no more than this amount to its environment. Also, the system dissipates energy at the "rate" $r_o u^2(k)$. In [13], [17] this is referred to as "input strictly passive." When the coefficient p satisfies $p \geqslant p_o > 0$, we have dissipation at the "rate" $p_o y^2(k)$, called "output strictly passive" in [13], [17]. Connections to other I/O descriptions, e.g., sector or conic section theory [22], [23] are straightforward. These I/O descriptions do not consider the behavior of the state trajectory, setting them apart from the notions of hyperstability, dissipativity, and π-sharing.

A hyperstable system [3], [7] can be considered a Lyapunov stable system whose state remains bounded when driven by inputs from a certain class. This class of inputs (see (A-4) in the Appendix) depends on the state trajectory and is often expressed as due to feedback from the system output.

Lemma 2: S is hyperstable [strictly hyperstable] if there exists an $\alpha > 0$ such that S is π-sharing with respect to (Γ, Q, p, r) where $\Gamma(k) \geqslant \alpha I$, $\forall k$ [$Q(k) \geqslant \alpha I$, $\forall k$], and $p(k), r(k) \geqslant 0$, $\forall k$.

Proof: See the Appendix. △△△

Similar to the hyperstability results, the following stability results are available via π-sharing. It will be convenient to collect I/O and state stability notions in the following.

Definition: A system S is *π-stable* if

a) there exist γ_1, γ_2 such that

$$\|y\|_T \leqslant \gamma_1\|u\|_T + \gamma_2|x(0)|, \qquad \forall\, u \in \mathbb{L},\ \forall\, x(0) \in \mathbb{R}^n,\ \forall\, T \in \mathbb{N},$$

and if

b) there exist γ_3, γ_4 such that

$$\max_{k \leqslant T} |x(k)| \leqslant \gamma_3 \|u\|_T + \gamma_4 |x(0)|, \quad \forall\, u \in \mathbb{L},\ \forall\, x(0) \in \mathbb{R}^n,\ \forall\, T \in \mathbb{N}.$$

Remark: Condition a) is a finite input–output gain [13] result for S, which implies that S is $\mathbb{l}^2$-stable [1]. This can also be seen as a "finite mean-square gain" condition for S. The coefficient γ_1 is the "gain" of the input/output mapping. Condition b) yields a bound on the state in terms of the summed-squared value of the input and the norm of the initial state. Thus, b) implies the Lyapunov stability of the zero solution when $u = 0$. The simultaneous dependence on $\|u\|_T$ and $|x(0)|$ in a) and b) results in a stronger form of stability than the statement "S is $\mathbb{l}^2$-stable and Lyapunov stable." This stability notion differs from the hyperstability formulation in the class of inputs considered (it does not depend on the state trajectory), and in the fact that explicit bounds on the norms of y and x are available in terms of the norm of u.

In keeping with the notion of strict hyperstability, where the state converges to zero, a definition of strict π-stability, where

$$\|x\|_T \leqslant \gamma_5 \|u\|_T + \gamma_6 |x(0)|, \qquad \forall\, T \in \mathbb{N},\ \forall\, u \in \mathbb{L},\ \forall\, x(0) \in \mathbb{R}^n$$

for some γ_5, γ_6, can also be useful [21]. However, this extension will not be explored in this paper.

The concepts of π-sharing and π-stability are linked by the following result.

Theorem 1: Let S be π-sharing with respect to (Γ, Q, p, r).

i) If there exist $r_o \in \mathbb{R}$, $p_o > 0$ such that

$$p(k) \geqslant p_o,\ r(k) \geqslant r_o,\ \forall\, k$$

then S has finite mean square gain as given by definition a) for π-stability.

ii) If in addition to i), there exists $\gamma > 0$ such that

$$\Gamma(k) \geqslant \gamma I, \qquad \forall\, k$$

then S is π-stable.

Proof: See the Appendix. △△△

These π-sharing results thus far are most similar to those obtained from the dissipative systems approach [10], [15], [18]. The dissipative results for single systems and certain interconnections are actually more general than considered here in the cases of purely Lyapunov (no inputs) or purely I/O (no consideration of state) stability studies. The advantage of Theorem 1 over the dissipativity results is that state and I/O stability are considered *simultaneously*, more fully exposing the effects of initial states and inputs on the behavior of the output and state trajectory. This distinction is most important when considering nonlinear systems, as superposition of the I/O and state stability results is not possible. Also, the dissipative approach is characterized by detectability *assumptions*, which are often invalid for the parameter identification algorithms studied in Section III. The uniform positive definite condition on Γ in Theorem 1 can be a weaker and more easily checked condition for these algorithms.

Pi-Sharing Results for Interconnections

Based on the linear energy supply relation (2-5), the following result shows that the π-sharing properties of the feedback interconnection depend on simple conditions concerning the π-coefficients (Γ, Q, p, r).

Theorem 2 (Feedback Connection): Let S_1 and S_2 be π-sharing with respect to $(\Gamma_1, Q_1, p_1, r_1)$ and $(\Gamma_2, Q_2, p_2, r_2)$, and let S be the system resulting from the feedback connection of S_1 and S_2 (Fig. 1). If

$$r_1(k) + p_2(k) > 0,\ \forall\, k \tag{2-11}$$

then S is π-sharing with respect to

$$\Gamma = \begin{bmatrix} \Gamma_1 & O \\ O & \Gamma_2 \end{bmatrix} \qquad Q = \begin{bmatrix} Q_1 & O \\ O & Q_2 \end{bmatrix} \tag{2-12}$$

$$r = \frac{r_1 p_2}{r_1 + p_2} \qquad p = p_1 + r_2. \tag{2-13}$$

(Γ, Q, p, r) are called composite π-coefficients for the interconnection S.

Proof: See the Appendix. △△△

Using Theorem 2 with the results of Theorem 1 and Lemmas 1 and 2 we see that a feedback connection of π-sharing systems, at least formally, may be passive, hyperstable, or π-stable without requiring both subsystems to have these properties. For example, it is not necessary in (2-13) for r_1, p_1, r_2, and p_2 to all be positive in order that r and p are positive. Also, a system may be shown π-stable without being passive or hyperstable according to Lemmas 1 and 2, since r is not required to be positive for π-stability. This emphasizes that "passivity" and "stability" are distinct concepts.

The direct connection between y_1 and u_2 in Fig. 1 leads to the form of the composite p in (2-13). Compensation for energy generation in one subsystem (say $r_2 < 0$) is possible if the other subsystem has "excess dissipation," given by $p_1 > 0$. This sharing is directly reflected in the composite coefficient p. The indirect connection between u_1 and y_2, however, results in a rather indirect form of sharing due to the form of r in (2-13), since $r \leqslant \min(r_1, p_2)$.

While the *stability* results given here could have been derived from the dissipative approach rather than via π-sharing (the overall conditions on r_1, p_1, r_2, and p_2 turn out to be the same), some subtle differences appear when considering the energy *dissipative* properties of a feedback interconnection. These differences stem from the interconnection structures considered. In this context, the dissipative approach [13], [17] considers the feedback structure as a connection of two *one-ports* into a single *two-port* system (two inputs, two outputs). The π-sharing approach considers the resulting connection as a single *one-port* (Fig. 1). The energy dissipative properties of the two-port cannot always be parameterized with respect to a single input and output as would be anticipated in simplifying the two-port to a one port (i.e., setting one input to zero). Essentially, the π-sharing definition requires the dissipation properties of a system to be expressible in terms of the state and the actual variables of interconnection. At least for the parameter identification algorithms considered here, this specialized view provides the more direct analysis.

The basic hyperstability result for interconnections [24] requires both subsystems to be hyperstable in order that the composite system is hyperstable. Thus, π-sharing, via Theorem 2, affords an extension through sharing in the r and p coefficients, much like that available with the generalized passivity [2], [22] and dissipative [13], [17] approaches. In, e.g., [4] various loop transformations are employed to obtain sharing-like hyperstability results, although some fundamental distinctions between these various approaches remain. Further discussion is provided in the remarks following Theorem 3 below.

Pi-Coefficient Computation

For the simple conditions on the subsystems in a feedback connection (Theorem 2) to be useful in design and analysis, some means of finding "tight" sets of π-coefficients for the subsystems is necessary. As foreshadowed by [22], such simplified approaches to the analysis of complicated systems can provide important information, but necessary conditions for, e.g., stability are often unavailable. An algebraic technique for finding "tight" sets of π-coefficients in terms of a state-space representation of S is given by:

Theorem 3: Suppose the functions f and h in the system description (2-4) are such that for some functions A, B, C, d, possibly depending on k, u, or x (of which we suppress all but k in the notation), we can write (2-4) equivalently as

$$x(k+1)=A(k)x(k)+B(k)u(k) \tag{2-14}$$

$$y(k)=C^T(k)x(k)+d(k)u(k). \tag{2-15}$$

Then S is π-sharing with respect to the scalar sequences r and p and symmetric p.s.d. matrix sequences Γ and Q if

$$M(k)=\begin{bmatrix} M_1(k) & M_2(k) \\ M_2^T(k) & M_4(k) \end{bmatrix} \tag{2-16}$$

is negative semidefinite (n.s.d.) for all k, where

$$M_1(k)=A^T(k)\Gamma(k+1)A(k)-\Gamma(k)+p(k)C(k)C^T(k)+Q(k) \tag{2-17}$$

$$M_2(k)=A^T(k)\Gamma(k+1)B(k)-\frac{1}{2}C(k)+p(k)d(k)C(k) \tag{2-18}$$

$$M_4(k)=B^T(k)\Gamma(k+1)B(k)-d(k)+r(k)+p(k)d^2(k). \tag{2-19}$$

The functions (Γ, Q, p, r) are then π-coefficients for S.

Proof: See the Appendix. △△△

Remarks: i) The specialization of (2-14) and (2-15) from (2-4) allows the conditions to be stated in terms of properties of a matrix of functions $M(k)$. Even though the elements of $M(k)$ may be unknown, their structural interrelation may allow $M(k)$ to be shown n.s.d. as required by Theorem 2. Such is the case in the parameter identification algorithm analysis of Section III. More general system forms lead to conditions on sets of nonlinear equations of matrix valued functions [15].

ii) In continuous time, differentiability of the state trajectory and the "storage function" $\Phi(x)$ ($x^T\Gamma x$ here) together with reachability of the state space $\mathbb{R}^n$ allows a converse to this theorem [15]. In general, the converse does not hold in discrete time, even with standard reachability assumptions. However, it is necessary that $M(k)$ have nonpositive eigenvalues "along the trajectories" $[x^T(k), u(k)]$. (See the proof in the Appendix.) As a practical matter, the state trajectories will be unknown for all but the simplest nonlinear time-varying systems, and only the sufficient condition given in the theorem will be useful.

iii) For linear time invariant systems, the $M(k)$ matrix can be simplified (set $p(k) = r(k) = 0$) to obtain the familiar Kalman–Popov–Yakubovich equations [8] for establishing the positive reality of the system (2-14), (2-15). In the hyperstability approach of [4], the $p(k)$ terms in $M(k)$ are obtained by block diagram manipulation of a feedback gain around the system. Similarly, the $r(k)$ term in (2-19) can be obtained by introducing an external feedforward gain. Thus, the application of Theorems 1 and 2 to a feedback system can be interpreted as a series of additive loop transformations to obtain a new feedback system composed of passive (or hyperstable) subsystems. However, the stability properties of this new system are not in general the same as those of the original feedback system. Additional conditions can provide the desired equivalence with respect to stabilty, e.g., [1], but such conditions must be verified in each case. Since this transformation is absorbed in the application of Theorems 1 and 2 in a way that retains the original feedback system signals, there are no additional conditions to check in the π-sharing approach.

iv) By the structure of $M(k)$, if it is n.s.d. for positive $r(k)$ and $p(k)$, then it is n.s.d. with $r(k) = p(k) = 0$. In the case where $M(k)$ corresponds to a linear time invariant system, the system is then passive (i.e., positive real) even when subjected to the (positive) feedback gain p and the (negative) feedforward gain r.

A Simple Example

Together with Lemmas 1 and 2, and Theorems 1 and 2, Theorem 3 provides an algebraic means of determining the passivity, hyperstability, and π-stability of a system or feedback interconnection of systems. The parameter identification algorithm of the next section is an example of a nontrivial application of these tools in determining stability. At this point, though, a simpler example is more useful in illustrating the points discussed above.

Example: Consider the feedback connection S of the systems S_1 and S_2 as in Fig. 1, where S_1 and S_2 are described by

$$S_1\begin{cases} x_1(k+1)=\frac{1}{2}x_1(k)+u_1(k) \\ y_1(k)=x_1(k)+2u_1(k) \end{cases} \tag{2-20}$$

$$S_2\begin{cases} x_2(k+1)=0x_2(k)+u_2(k) \\ y_2(k)=0x_2(k)+d(k)u_2(k). \end{cases} \tag{2-21}$$

We wish to choose the parameter $d(k)$ in S_2 so the feedback interconnection is "well-behaved." Note that in an input–output sense, S_2 is simply the gain $d(k)$. If $d(k)$ were fixed, then we could apply z-transform techniques to obtain the $\mathfrak{l}^p$-stability [1] of S, $1 \leqslant p \leqslant \infty$, for $d < -3/4$ or $d > -1/4$. If we allow d to vary with time, the π-sharing techniques still apply while the transform techniques do not. To find π-coefficients for S_1 and S_2, we use Theorem 3. It can easily be checked that

$$(\Gamma_1, Q_1, p_1, r_1)=(1/4, 0, 3/16, 1) \tag{2-22}$$

provides that $M(k) \equiv O$ (hence, is n.s.d.) for S_1 and that

$$(\Gamma_2, Q_2, p_2, r_2)=(\epsilon, \epsilon, 0, d-\epsilon) \tag{2-23}$$

yields $M(k) \equiv O$ for S_2, for any $\epsilon > 0$. Using Theorem 2 we find composite coefficients for the interconnection S:

$$\Gamma=\begin{bmatrix} 1/4 & 0 \\ 0 & \epsilon \end{bmatrix}, \quad Q=\begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix} \tag{2-24}$$

$$r=\frac{0}{1+0}=0, \quad p=3/16+d-\epsilon. \tag{2-25}$$

Now $p(k)$ is bounded away from zero and positive if $d(k)$ is bounded away from $-3/16$. Also, $\Gamma(k)$ is uniformly positive definite, $Q(k)$ is p.s.d., $\forall k$, and $r(k) \equiv 0$. By Lemmas 1 and 2 and Theorem 1 we have that S is passive, hyperstable, and π-stable with the lower bound of $-3/16$ on $d(k)$. The π-stability of S implies it is $\mathfrak{l}^2$-stable and the composite state is bounded whenever the input is in $\mathfrak{l}^2$.

Compared to the z-transform approach, when $p(k)$ is fixed, the π-sharing approach has both different conditions [e.g., on $d(k)$] and different results. General conclusions about the relative strengths of these two approaches are therefore difficult to obtain. Interesting comparisons to specialized input–output stability approaches, such as the circle [1], [23] and Popov [1], [24] criteria could also be made, but we will not pursue these here. However, this example demonstrates some key features of the π-sharing approach claimed earlier. By definition of passivity and hyperstability (see the proofs of Lemmas 1 and 2), it is easy to see the subsystem S_2 is neither passive nor hyperstable when $d(k) < 0$, $\forall k$. However, the composite system is both passive and hyperstable as long as $d(k) \geqslant -3/16 + \epsilon$, for some $\epsilon > 0$. Note also that when $d(k)$ is bounded, S_2 is π-stable, according to the definition, so a system may, in fact, be π-stable *without* being passive or hyperstable. The next section provides a more meaningful demonstration of the usefulness of the π-sharing approach in stability analysis.

III. Application: Recursive Parameter Identification

We consider the problem of recursively identifying the parameters of a linear time-invariant plant P described by

$$y(k+1)=\theta^T\psi(k) \tag{3-1}$$

where

$$\theta^T=[a_1, \cdots, a_n, b_1, \cdots, b_m] \tag{3-2}$$

$$\psi^T(k)=[y(k), \cdots, y(k-n+1), u(k), \cdots, u(k-m+1)] \tag{3-3}$$

are *plant parameter* and *plant information vectors,* respectively. The plant output is predicted using *a posteriori* parameter estimates in a parallel (output error) structure

$$\hat{y}(k+1)=\hat{\theta}^T(k+1)\phi(k) \tag{3-4}$$

where the *predictor information vector* is

$$\phi^T(k)=[\hat{y}(k), \cdots, \hat{y}(k-n+1), u(k), \cdots, u(k-m+1)]. \tag{3-5}$$

If we define the *prediction error* $v(k)$ and *equation error* $e(k)$ according to

$$v(k+1) \triangleq y(k+1)-\hat{y}(k+1)+\eta(k+1) \tag{3-6}$$

$$e(k+1) \triangleq [\theta^T-\hat{\theta}^T(k+1)]\phi(k) \triangleq \tilde{\theta}^T(k+1)\phi(k) \tag{3-7}$$

where η is an additive output disturbance, the parallel prediction structure yields the following relation between e, v, and η [6]:

$$v(k+1)=\sum_{i=1}^{n} a_i v(k+1-i) + e(k+1)+\eta(k+1)-\sum_{i=1}^{n} a_i\eta(k+1-i). \tag{3-8}$$

In polynomial shift operator (q^{-1}) notation, (3-8) is written as

$$v(k+1)=\frac{1}{A(q^{-1})}\{e(k+1)+w(k+1)\} \tag{3-9}$$

where $A(q^{-1}) = 1 - \Sigma_{i=1}^{n} a_i q^{-i}$ and $w(k + 1) \triangleq A(q^{-1})\{\eta(k + 1)\}$. This purely autoregressive system has a convenient state-space realization for use in Theorem 3. We denote this as S_1

$$S_1\colon\ x_1(k+1)=A_1x_1(k)+B_1[e(k+1)+w(k+1)] \tag{3-10}$$

$$v(k+1)=C_1^Tx(k)+d_1[e(k+1)+w(k+1)] \tag{3-11}$$

where

$$A_1=\begin{bmatrix} a_1 & \cdots & a_n \\ 1 & & 0 \\ & \ddots & \vdots \\ O & 1 & 0 \end{bmatrix};\ B_1=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3-12}$$

$$C_1^T=[a_1, \cdots, a_n];\ d_1=1. \tag{3-13}$$

The parameters in the parallel predictor are updated according to

$$\hat{\theta}(k+1)=\hat{\theta}(k)+h(k)P(k)\phi(k)v(k+1) \tag{3-14}$$

where $h(k)$ and $P(k)$ are the (as yet unspecified) scalar "step size" and "direction matrix." By subtracting θ from both sides and adjoining the equation error (3-7), we form a state-space representation for the relation between $v(k + 1)$ and $-e(k + 1)$ which we denote as S_2

$$S_2\colon\ \tilde{\theta}(k+1)=A_2\tilde{\theta}(k)+B_2(k)v(k+1) \tag{3-15}$$

$$-e(k+1)=C_2^T(k)\tilde{\theta}(k)+d_2(k)v(k+1) \tag{3-16}$$

where

$$A_2=I;\ B_2(k)=-h(k)P(k)\phi(k) \tag{3-17}$$

$$C_2^T(k)=-\phi^T(k),\ d_2(k)=h(k)\phi^T(k)P(k)\phi(k). \tag{3-18}$$

S_2 is obviously time-varying, and since the entries of B_2, C_2, and d_2 depend on past values of $\hat{\theta}$ and v, S_2 is also nonlinear. As an aside, we note that (3-14) is an implicit expression for $\hat{\theta}(k + 1)$. It can be solved explicitly for $\hat{\theta}(k + 1)$ for implementation [12], but the implicit form is more useful for analysis.

In what has become a standard approach [4], [6], we examine the behavior of this class of parameter identification algorithms by noting that S_1 and S_2 are interconnected in a feedback configuration S (error model) as in Fig. 1, but with composite input w and output v. Convergence of the prediction error v to zero would allow persistent excitation conditions on u to be given such that ϕ is also persistently exciting [9]. The entire error model S could then be shown asymptotically stable, implying that the parameter errors $\tilde{\theta}$ converge to zero. The π-sharing approach to the stability of S addresses the convergence of the prediction error v, using sharing between S_1 and S_2 to provide the π-stability of S. Sharing exposes a relationship between the ubiquitous SPR assumption on S_1, as in, e.g., [4], and the "power" in the plant input u sufficient for stability as given in [5], [12].

First we use Theorem 3 to find π-coefficients for S_1 and S_2.

Lemma 3: For S_1, let $\alpha = C_1^TC_1 \triangleq \Sigma_{i=1}^{n} a_i^2$, then S_1 is π-sharing with respect to (Γ_1, Q_1, p_1, r_1) given by

$$\Gamma_1(k) \equiv \begin{bmatrix} n\alpha & & O \\ & (n-1)\alpha & \\ & & \ddots & \\ O & & & \alpha \end{bmatrix};\ Q_1(k) \equiv \frac{\alpha}{2}I \tag{3-19}$$

$$p_1(k) \equiv 1/2-n\alpha;\ r_1(k) \equiv 1/2. \tag{3-20}$$

Proof: Direct substitution shows that $M(k)$ is block diagonal, M_1 has nonpositive eigenvalues, and $M_4 = 0$ so Theorem 3 holds. △△△

Lemma 4: For S_2, if $P^{-1}(0)$ is symmetric p.d. and $P(k)$ is updated according to

$$P^{-1}(k+1)=\gamma(k)P^{-1}(k)+\delta(k)\phi(k)\phi^T(k) \tag{3-21}$$

where $0 < \gamma(k) \leqslant 1$, $\forall k$, and $0 \leqslant \delta(k)$, $\forall k$, and if the step size is given by $h^{-1}(k) = 2\gamma(k)$, then S_2 is π-sharing with respect to (Γ_2, Q_2, p_2, r_2) where

$$\Gamma_2(k)=P^{-1}(k);\ Q_2(k)=(1-\gamma(k))P^{-1}(k) \tag{3-22}$$

$$p_2(k)=-\delta(k);\ r_2(k)=\frac{\phi^T(k)P(k)\phi(k)}{4\gamma(k)}. \tag{3-23}$$

Proof: By direct substitution, $M(k)$ is identically zero. The constraints on the update for $P(k)$ provide that $\Gamma_2(k)$ and $Q_2(k)$ are p.s.d., $\forall k$. △△△

Remark: The proofs of Lemmas 3 and 4 adopt the strategy of requiring $M(k)$ to be block diagonal. While this greatly simplifies the task of showing it to be n.s.d., this prevents retaining greatest generality in the proof. Other algorithms may require a more general use of Theorem 3 than that demonstrated here.

By (3-22) and (3-23), the parameters γ and δ influence the energy dissipative properties of S_2; that is, the properties of the *parameter estimate update* portion of the adaptive algorithm. Note that S_2 may generate energy if $\delta > 0$. The subsystem S_1, and thus the dissipation properties parameterized by α in (3-19) and (3-20), are determined by the structure of the *predictor* portion of the adaptive algorithm. S_1 also may generate energy if $1/2 - n\alpha < 0$. The π-sharing approach provides the following characterization of the adaptive algorithm as a whole in terms of these parameters and the properties of the information vector sequence ϕ.

Theorem 4: For the general output error algorithm described by (3-1) to (3-7), (3-14), and (3-21), if

C1: there exists an $\epsilon_1 > 0$ such that

$$\frac{1}{2} - \delta(k) \geqslant \epsilon_1, \qquad \forall\, k$$

C2: and there exists an $\epsilon_2 > 0$ such that

$$\frac{1}{2} - n \sum_{i=1}^{n} a_i^2 + \frac{\phi^T(k)P(k)\phi(k)}{4\gamma(k)} \geqslant \epsilon_2, \qquad \forall\, k$$

then the prediction error is bounded in the sense

R1: there exists γ_1, γ_2 such that

$$\|v\|_T \leqslant \gamma_1 \|w\|_T + \gamma_2 |x(0)|, \qquad \forall\, T \in \mathbb{N},\ \forall\, x(0) \in \mathbb{R}^n.$$

If in addition to C1 and C2

C3: there exists $\alpha > 0$ such that

$$P^{-1}(k) \geqslant \alpha I, \qquad \forall\, k$$

then the parameter error is bounded in the sense

R2: there exist γ_3, γ_4 such that

$$\max_{k \leqslant T} |\tilde{\theta}(k)| \leqslant \gamma_3 \|w\|_T + \gamma_4 \left[|\tilde{\theta}(0)| + \sum_{i=1}^{n} |v(-i+1)| \right]$$

$$\forall T \in \mathbb{N},\ \forall\, \tilde{\theta}(0) \in \mathbb{R}^n,\ \forall\, v(-i+1) \in \mathbb{R},\ i \in [1,\ n].$$

Proof: By Theorem 2, condition C1 implies the error system S corresponding to the algorithm is π-sharing with respect to

$$r(k) = \frac{-\delta(k)}{1 - 2\delta(k)},$$

$$p(k) = \frac{1}{2} - n\alpha + \frac{\phi^T(k)P(k)\phi(k)}{4\gamma(k)}$$

with $\alpha = \Sigma_{i=1}^{n} a_i^2$. By Theorem 1, part i, condition C2 yields the I/O stability statement R1. Condition C3 in Theorem 1, part ii, yields the π-stability of S, which specializes to the statement R2. △△△

Conditions C1 and C2 ensure that energy generation in one subsystem is overcome by excess dissipation in the other. C1 is easily satisfied since the parameter δ is designer-selected. Condition C2 is the key condition demonstrating the advantages of sharing for this parameter identification example. It requires that the composite coefficient $p(k)$ is positive and bounded away from zero. This can be accomplished in two different ways. First, if the coefficients of the plant denominator polynomial a_i are small enough by $\Sigma_{i=1}^{n} a_i^2 < 1/2n$, then α may be chosen so that $1/2 - n\alpha > 0$. Since $P(k)$ is p.s.d. and $\gamma(k) > 0$, $p(k)$ is bounded away from zero without utilizing the sharing available from the $\phi^T P \phi$ term in C2. This is the essential effect of requiring S_1 to be strictly positive real (SPR). (Note by Lemma 1, this condition on the a_i implies that S_1 is strictly passive, hence strictly positive real [7], [11].) This approach captures the motivation behind common schemes of sidestepping the SPR condition, e.g., [25]–[27]. They seek to change the algorithm in order to modify the operator subject to the SPR condition.

The second approach to obtain a positive composite $p(k)$ (condition C2) exploits the sharing available from the $\phi^T P \phi$ term by requiring

$$\frac{\phi^T(k)P(k)\phi(k)}{4\gamma(k)} > n\alpha, \qquad \forall\, k \tag{3-24}$$

by choice of the u components of $\phi(k)$ and the scalars $\gamma(k)$ and $\delta(k)$ affecting the update (3-21) for $P(k)$. The result of [5] can be easily cast in this form, since there $\gamma(k) \equiv 1$ and $\delta(k) \equiv 0$ which results in $P(k) \equiv P(0)$. If the plant is only known to be stable, then an upper bound on α can be chosen for which a lower bound on $\Sigma_{i=1}^{m} u^2(k-1)$ may be stated causing the inequality (3-24) to be satisfied. Since in many applications the plant input u is easily influenced, this condition is generally much more practical than an SPR assumption on the plant under identification. Note this is fundamentally different from a persistent excitation condition on u [9]; it is really more of a "persistent power" condition. See [14] for more discussion of "persistent power" sequences. With $P(k)$ a fixed p.d. matrix, the composite coefficient $\Gamma(k)$ is uniformly positive definite, satisfying C3 of Theorem 4. In [5], the error system input is zero, so it certainly belongs to $\mathbb{l}^2$, hence, the output $v \in \mathbb{l}^2$ and the composite state $x = [x_1^T, x_2^T]^T$ is bounded. This implies the prediction error converges to zero and the parameter errors are bounded.

Thus, in this second case, we see an advantage in exploiting the sharing possible between systems as completely as possible, rather than making extra (SPR) assumptions on the systems themselves. This tradeoff in condition C2 also suggests that a mixture of SPR and persistent power conditions, given by the relative magnitudes of the a_i and $\phi^T P \phi$ terms, could be used to obtain more relaxed conditions for algorithm stability in some applications. Since the π-stability property of S holds for any error system input in $\mathbb{L}$, we have extended the results of [4], [5], obtaining the $\mathbb{l}^2$-stability and the finite mean-square gain stability of the disturbance-to-prediction error relation. These results are not available using the hyperstability approach of [4], [5].

The π-sharing approach also allows the sharing phenomenon to be extended to more general classes of algorithms than considered in [5]. The new results of [12], for example, are based on condition C2 of Theorem 4, but various other forms of the $P(k)$ update within (3-21) are considered. Essentially, whenever $\phi^T P \phi$ can be set uniformly large enough to compensate for non-SPR deficiencies in S_1, by selection of the input sequence u, then persistent power conditions may be used in place of SPR conditions.

Given the π-stability results, stronger bounded input-bounded output stability results may be obtained using linearization techniques and exponential stability, e.g., [19]. With the addition of a persistent excitation condition on ϕ, the $\mathbb{l}^\infty$-"gain" of the error system may be derived [20], resulting in a "local" bounded disturbance-bounded prediction error-bounded parameter error stability result for the algorithm. The result is in general "local," since large disturbances may destroy the linearization and persistent excitation properties used to derive the result.

It is important to note that the persistent excitation conditions are on the information vector ϕ, some of whose elements depend on the evolution of the parameter estimates, which are not known *a priori*. Transferring this condition to one on the input sequence u alone (as done for the persistent power condition in [5], [12]) is a key step toward providing persistent excitation. This step uses linearization, which depends critically on the convergence of the prediction error [9]. The stability results in this paper, using a tradeoff between SPR and persistent power conditions, allow the results of [9] to be extended to include the case of disturbances in $\mathbb{l}^2$, and to provide closed-loop I/O gains. Thus, the π-sharing stability results provide a vital basis upon which to derive more general bounded input-bounded output stability results.

IV. Conclusion

The definition of π-sharing provides a useful characterization of the behavior of systems connected in feedback. This characterization is embodied in a set of π-coefficients (functions) pertaining to the system "energy supply," and in terms of these coefficients close ties exist to the well-known notions of passivity, hyperstability, and dissipativity—as well as to a stability notion introduced here as π-stability. An algebraic test on a system state-space representation exists which is sufficient to determine π-coefficients, and in these terms there are explicit expressions for the sharing possible between interconnected systems.

The use of π-sharing in stability analysis was illustrated by two examples. The first provided some comparisons between passivity, hyperstability, π-sharing, and π-stability for a simple feedback system. The second example presented a general parameter identification algorithm as a nonlinear time-varying feedback system for which the algebraic conditions for π-sharing could be satisfied. The π-sharing approach for the second example provided a straightforward explanation in terms of sharing for the differences between the results of [4] and [5] with regard to the SPR assumptions. Also we found that the shift in viewpoint away from hyperstability provided results of a related but essentially different nature—more in line with input–output stability results. The π-sharing stability results also form an important basis for deriving conditions for bounded disturbance-bounded prediction error stability.

The π-sharing approach shows promise in treating various other "adaptive systems" due to the structure in the $M(k)$ matrix (Theorem 3) given by the particular forms of the parameter estimate update for these algorithms. This algebraic tool provides a useful approach for such dynamical systems, where explicit solutions are unavailable. For some parameter identification algorithms, e.g., those using information vector filtering, the I/O "gains" provided by the π-sharing approach are instrumental [21] in obtaining overall stability results.

Appendix

Proof of Lemma 1: From [1], a system is passive [strictly passive] by definition if there exists a $\delta \geqslant 0$ [$\delta > 0$] and a $\beta \in \mathbb{R}$ such that

$$\langle u, y\rangle_T \geqslant \delta\|u\|_T^2+\beta, \qquad \forall\, u \in \mathbb{L},\ \forall\, T \in \mathbb{N}. \tag{A-1}$$

From the definition of π-sharing we have

$$\langle u, y\rangle_T \geqslant (p)\|y\|_T^2+(r)\|u\|_T^2-(\Gamma)|x(0)|^2, \quad \forall\,(u, \cdot, y)\in C_S,\ \forall\, T\in\mathbb{N} \tag{A-2}$$

but since the systems considered are mappings defined for all $u \in \mathbb{L}$, by taking $\beta = -(\Gamma)|x(0)|^2$ the hypotheses on r and p yield

$$\langle u, y\rangle_T \geqslant r_o\|u\|_T^2+\beta, \qquad \forall\, u\in\mathbb{L},\ \forall\, T\in\mathbb{N} \tag{A-3}$$

and the result follows. △△△

Proof of Lemma 2: From [3], [7], a system is hyperstable by definition if whenever the input u lies in the class B, defined for $\delta > 0$ [possibly depending on $x(0)$] by

$$B=\{u\in\mathbb{L}\,|\,\langle u, y\rangle_T \leqslant \delta \max_{k\leqslant T} |x(k)|, \qquad \forall \leqslant T\in\mathbb{N}\}, \tag{A-4}$$

then the state remains bounded in the sense that there exist M and σ such that

$$|x(k)| \leqslant \sigma|x(0)|+M, \qquad \forall\, k. \tag{A-5}$$

If in addition, for bounded $u \in B$, $\lim_{k\to\infty} |x(k)| = 0$, then the system is said to be strictly hyperstable.

By hypothesis we have

$$\langle u, y\rangle_T \geqslant (\Gamma)|x(T+1)|^2-(\Gamma)|x(0)|^2+(Q)\|x\|_T^2, \quad \forall\,(u, x, y)\in C_S, \qquad \forall\, T\in\mathbb{N}. \tag{A-6}$$

Since there exists a solution triple for S for every $u \in \mathbb{L}$, and B is contained in $\mathbb{L}$, if $\Gamma(k) \geqslant \alpha I$, $\forall k$, we have for every $u \in B$

$$\delta \max_{k\leqslant T+1} |x(k)| \geqslant \delta \max_{k\leqslant T} |x(k)| \geqslant \langle u, y\rangle_T \geqslant \alpha|x(T+1)|^2-(\Gamma)|x(0)|^2, \qquad \forall\, T\in\mathbb{N}. \tag{A-7}$$

Now if $\max_{k\leqslant T+1} |x(k)| > |x(T + 1)|$, $\forall T \in \mathbb{N}$, then $|x(k)|$ is bounded as claimed, so suppose for some T $\max_{k\leqslant T+1} |x(k)| = |x(T + 1)|$. Then the inequality (A-7) becomes

$$\delta|x(T+1)| \geqslant \alpha|x(T+1)|^2-(\Gamma)|x(0)|^2. \tag{A-8}$$

By the quadratic formula

$$|x(T+1)| \leqslant \frac{\delta}{2\alpha}+\frac{\sqrt{\delta^2+4\alpha(\Gamma)|x(0)|^2}}{2\alpha}. \tag{A-9}$$

Since this bound is independent of T, it follows that $|x(k)|$ is bounded as required, and the system is hyperstable. If instead $Q(k) \geqslant \alpha I$, $\forall k$, then we have

$$\delta \max_{k\leqslant T} |x(K)| \geqslant \alpha\|x\|_T^2-(\Gamma)|x(0)|^2 = \alpha\sum_{k=0}^{T}|x(k)|^2-(\Gamma)|x(0)|^2, \qquad \forall\, T\in\mathbb{N}. \tag{A-10}$$

By similar reasoning as above, this implies that $\Sigma|x(k)|^2$ is bounded, implying that $\lim_{k\to\infty} |x(k)| = 0$, and the system is strictly hyperstable. △△△

Proof of Theorem 1: i) By hypothesis

$$\langle u, y\rangle_T \geqslant (\Gamma)|x(T+1)|^2-(\Gamma)|x(0)|^2+(Q)\|x\|_T^2 +p_o\|y\|_T^2-\delta\|u\|_T^2, \qquad \forall\,(u, x, y)\in C_S,\ \forall\, T\in\mathbb{N} \tag{A-11}$$

where $\delta = |\min(0, r_o)|$. Since Γ and Q are p.s.d., using the Schwartz inequality, we obtain

$$\|u\|_T \cdot \|y\|_T \geqslant p_o\|y\|_T^2-\delta\|u\|_T^2-(\Gamma)|x(0)|^2 \quad \forall\,(u, x, y)\in C_S,\ \forall\, T\in\mathbb{N}. \tag{A-12}$$

Since $p_o > 0$, the quadratic formula yields

$$\|y\|_T \leqslant \left(\frac{1+\sqrt{p_o\delta}}{p_o}\right)\|u\|_T+(\sqrt{\gamma_o/p_o})|x(0)|, \quad \forall\, T\in\mathbb{N},\ \forall\, u\in\mathbb{L} \tag{A-13}$$

where γ_o = maximum eigenvalue of $\Gamma(0)$, and part (a) of the π-stability definition holds.

ii) Using (A-13) in (A-11) with the Schwartz inequality again, and since $p_o > 0$,

$$(\Gamma)|x(0)|^2+\|u\|_T^2\left(\delta+\frac{1+\sqrt{p_o\delta}}{p_o}\right)+\|u\|_T \cdot |x(0)|\sqrt{\gamma_o/p_o} \geqslant (\Gamma)|x(T+1)|^2+(Q)\|x\|_T^2,\ \forall\,(u, x, y)\in C_S,\ \forall\, T\in\mathbb{N}. \tag{A-14}$$

Let $\beta = \max(\gamma_o, \delta + (1 + \sqrt{p_o\delta})/p_o, \sqrt{\gamma_o/p_o})$, then

$$\beta(\|u\|_T+|x(0)|)^2 \geqslant (\Gamma)|x(T+1)|^2+(Q)\|x\|_T^2. \tag{A-15}$$

Now by hypothesis ii), and since $\|\cdot\|_T$ is monotone increasing in T, (A-15) yields

$$\lim_{k\leqslant T} |x(k)| \leqslant \sqrt{\beta/\gamma}(\|u\|_T+|x(0)|), \qquad \forall\, u\in\mathbb{L},\ \forall\, x(0)\in\mathbb{R}^n,\ \forall\, T\in\mathbb{N} \tag{A-16}$$

since the domain of S includes all of $\mathbb{L}$ for u and all of $\mathbb{R}^n$ for $x(0)$. Part b) of the π-stability definition follows.

Proof of Theorem 2: By the feedback structure and the π-sharing of S_1 and S_2

$$\begin{aligned}\langle u, y\rangle_T = \langle u_1, y_1\rangle_T + \langle u_2, y_2\rangle_T \geqslant (\Gamma_1)|x_1(T+1)|^2 - (\Gamma_1)|x_1(0)|^2 \\ + (\Gamma_2)|x_2(T+1)|^2 - (\Gamma_2)|x_2(0)|^2 + (Q_1)\|x_1\|_T^2 + (Q_2)\|x_2\|_T^2 \\ + (p_1+r_2)\|y\|_T^2 + (r_1)\|u_1\|_T^2 + (p_2)\|y_2\|_T^2 \\ \forall\, T\in\mathbb{N},\ \forall\,(u_1, x_1, y)\in C_{S_1},\ \forall\,(y, x_2, y_2)\in C_{S_2}. \end{aligned} \tag{A-17}$$

Define the composite system state $x^T = [x_1^T, x_2^T]$ so that with Γ, Q, and p as in the theorem statement

$$\langle u, y\rangle_T \geqslant (\Gamma)|x(T+1)|^2 - (\Gamma)|x(0)|^2 + (Q)\|x\|_T^2 + (p)\|y\|_T^2 + (r_1)\|u_1\|_T^2 + (p_2)\|y_2\|_T^2 \tag{A-18}$$

and we seek an r such that

$$(r_1)\|u_1\|_T^2 + (p_2)\|y_2\|_T^2 \geqslant (r)\|u_1+y_2\|_T^2 = (r)\|u\|_T^2 \tag{A-19}$$

holds for all u_1 in C_{S_1} and all y_2 in C_{S_2}. This is satisfied by the stronger condition

$$r_1(k)u_1^2(k) + p_2(k)y_2^2(k) \geqslant r(k)(u_1(k)+y_2(k))^2, \quad \forall\, u_1(k), y_2(k)\in\mathbb{R},\ \forall\, k\in\mathbb{N}. \tag{A-20}$$

Since $r_1(k) + p_2(k) > 0$, $\forall k$, divide both sides of (A-20) by $r_1(k) + p_2(k)$ to obtain

$$\lambda(k)u_1^2(k) + (1-\lambda(k))y_2^2(k) \geqslant \frac{r(k)}{r_1(k)+p_2(k)}(u_1(k)+y_2(k))^2 \tag{A-21}$$

for $\lambda(k) \triangleq r_1(k)/(r_1(k) + p_2(k))$. Note that (dropping the index k)

$$\begin{aligned}(\lambda u_1 - (1-\lambda)y_2)^2 = u_1^2(\lambda(\lambda-1)+\lambda) - 2\lambda(1-\lambda)u_1y_2 \\ + (1-\lambda-\lambda(1-\lambda))y_2^2 \geqslant 0 \quad \forall\, \lambda,\ u_1,\ y_2\in\mathbb{R}.\end{aligned} \tag{A-22}$$

Thus,

$$\lambda u_1^2 + (1-\lambda)y_2^2 \geqslant \lambda(1-\lambda)(u_1^2+y_2^2+2u_1y_2) \tag{A-23}$$

hence multiplying through by $r_1(k) + p_2(k)$ yields (A-20) with

$$r(k) = \frac{r_1(k)p_2(k)}{r_1(k)+p_2(k)}$$

satisfying the definition of π-sharing for the composite system S. △△△

Proof of Theorem 3: Using the state representation for S we have

$$\begin{aligned}(\Gamma)|x(k+1)|^2 - (\Gamma)|x(k)|^2 = [x^T(k),\ u(k)] \\ \cdot \begin{bmatrix} A^T(k)\Gamma(k+1)A(k) - \Gamma(k) & A^T(k)\Gamma(k+1)B(k) \\ B^T(k)\Gamma(k+1)A(k) & B^T(k)\Gamma(k+1)B(k)\end{bmatrix} \\ \cdot \begin{bmatrix} x(k) \\ u(k)\end{bmatrix}.\end{aligned} \tag{A-24}$$

Also

$$u(k)y(k) = [x^T(k),\ u(k)]\begin{bmatrix} O & \frac{1}{2}C(k) \\ \frac{1}{2}C^T(k) & d(k)\end{bmatrix}\begin{bmatrix} x(k) \\ u(k)\end{bmatrix} \tag{A-25}$$

and

$$p(k)y^2(k) = p(k)[x^T(k),\ u(k)] \begin{bmatrix} C(k)C^T(k) & C(k)d(k) \\ d(k)C^T(k) & d^2(k)\end{bmatrix}\begin{bmatrix} x(k) \\ u(k)\end{bmatrix} \tag{A-26}$$

so in summation from $k = 0$ to $k = T$ we have

$$\begin{aligned}\langle u, y\rangle_T = -\sum_{k=0}^{T}[x^T(k),\ u(k)]M(k)\begin{bmatrix} x(k) \\ u(k)\end{bmatrix} \\ + \sum_{k=0}^{T}[(\Gamma)|x(k+1)|^2 - (\Gamma)|x(k)|^2] \\ + \sum_{k=0}^{T}p(k)y^2(k) + \sum_{k=0}^{T}r(k)u^2(k) \\ + \sum_{k=0}^{T}x^T(k)Q(k)x(k), \\ \forall\,(u, x, y)\in C_S,\ \forall\, T\in\mathbb{N}\end{aligned} \tag{A-27}$$

where $M(k)$ is as given in (2-17)–(2-19). Thus, if $M(k)$ is n.s.d. $\forall k$, neglecting this term and using our notation for weighted norms (2-6)–(2-8) we have the desired inequality (2-9) for the π-sharing definition. △△△

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