Characterization of the noise and correlations in harmonically mode-locked lasers

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Received December 11, 2001; revised manuscript received April 16, 2002

In a harmonically mode-locked laser multiple optical pulses propagate inside the laser cavity. The noise in different pulses inside the laser cavity is in general correlated. Information regarding the sign and magnitude of the noise correlations is contained in the distribution of the spectral weight among the supermode noise peaks that appear in the pulse energy and timing noise spectral densities. We show that the supermode noise spectrum obtained experimentally by measurement of the photodetector current noise spectral density can be used to determine the correlations in the energy and the timing noise of different pulses in the laser cavity. We also present simple models for the timing noise in harmonically mode-locked lasers that demonstrate the relationship between the noise correlations and the supermode noise peaks. © 2002 Optical Society of America

OCIS codes: 140.4050, 270.2500, 320.5550.

1. INTRODUCTION

Harmonically mode-locked lasers are attractive as sources of high-repetition-rate optical pulses that can be used in electro-optic sampling, optical analog-to-digital conversion, optical telecommunication systems, and ultrafast optical measurements.[1–17] In a fundamentally mode-locked laser the active modulation is applied at the cavity round-trip frequency, and only a single optical pulse propagates inside the laser cavity. In a laser mode locked at the Nth harmonic, the active modulation is applied at a frequency N times the cavity round-trip frequency, and N different optical pulses propagate inside the laser cavity. The pulse repetition frequency is therefore N times the cavity round-trip frequency. Passive harmonic mode locking can also be accomplished with suitable laser cavity designs.[14–17]

The pulse energy and timing noise spectral density functions of mode-locked lasers can be obtained experimentally from the photodetector current noise spectral density by Von Der Linde's technique.[18] The noise spectral density functions of fundamentally mode-locked lasers have noise peaks at multiples of the pulse repetition frequency, and therefore there is only one noise peak in a bandwidth equal to the pulse repetition frequency. The noise spectral density functions of harmonically mode-locked lasers can have noise peaks at multiples of the cavity round-trip frequency in addition to the noise peaks at multiples of the pulse repetition frequency.[4–6,12,19–21] Therefore there can be N different noise peaks in a bandwidth equal to the pulse repetition frequency. The $N - 1$ additional noise peaks have been called the supermode noise peaks in the literature.[4–6,12,19–21] The supermode noise peaks have been related to the beating between different supermodes whose center frequencies differ by multiples of the cavity round-trip frequency.[12,19,20,21]

In this paper we show that the supermode noise peaks in the pulse noise spectral density functions are directly related to the correlations in the noise in different pulses inside the laser cavity. These noise correlations can arise in various ways, some of which are described in this paper. The nature of the correlations in the noise of different pulses inside the laser cavity provides insight into the underlying physics. Noise in pulses in a mode-locked laser can have contributions from several different sources, such as spontaneous emission, vacuum fluctuations, gain fluctuations, and radio frequency (RF) and oscillator noise.[22–24] These individual noise contributions can cause different types of noise correlation in the pulses in a harmonically mode-locked laser. For example, the contribution to the pulse timing noise from spontaneous emission and vacuum fluctuations is uncorrelated in different pulses, and the contribution to the pulse timing noise from the phase noise of the RF oscillator is positively correlated in all the pulses, since all the pulses in the cavity are driven by the same active modulator. Gain competition in harmonically mode-locked semiconductor lasers can cause negatively correlated energy fluctuations in different pulses. In addition, intercavity reflections that couple energy from one pulse to another can also cause the noise in the pulses to become correlated. Studying the correlations in the noise of different pulses can therefore provide valuable information about the dynamics inside the laser.

We show that the noise correlations among the pulses can be determined exactly from the experimentally measured pulse noise spectral density functions. The information regarding these correlations resides in the distribution of the spectral weight among the N different noise peaks (including the $N - 1$ supermode noise peaks) that appear in the noise spectral density functions in a bandwidth equal to the pulse repetition frequency. We also present simple models for the pulse timing noise in harmonically mode-locked semiconductor lasers that illustrate the points discussed above.

2. PULSE NOISE CORRELATION FUNCTIONS AND SPECTRAL DENSITY FUNCTIONS

In this section the correlation functions and the spectral density functions are defined for the noise in optical pulses coming out of a mode-locked laser. The expressions obtained given are applicable to both fundamentally and harmonically mode-locked lasers.

The average intensity $\langle I_p(t)\rangle$ of an optical pulse train coming out of a mode-locked laser is

$$\langle I_p(t)\rangle = \sum_n E_p f(t - nT), \tag{1}$$

where the angle brackets $\langle\,\rangle$ denote ensemble averaging. The average energy of a single optical pulse is E_p. The time-dependent intensity of a single pulse is given by the function $f(t)$, which is normalized such that its integral equals unity. The pulse repetition rate is $1/T$. If $\Delta e[n]$ and $\Delta t[n]$ are the energy and the timing fluctuations of the nth optical pulse, respectively, then in the presence of these fluctuations the intensity of the optical pulse train is

$$\begin{aligned} I(t) &= \sum_n (E_p + \Delta e[n]) f(t - nT - \Delta t[n]) \\ &\approx \sum_n (E_p + \Delta e[n]) f(t - nT) \\ &\quad - E_p \sum_n \Delta t[n] \frac{\mathrm{d}}{\mathrm{d}t} f(t - nT). \end{aligned} \tag{2}$$

Relation (2) assumes that the perturbations that distort the pulse shape are negligible. Assuming the noise in the pulse train to be stationary, the noise is completely characterized by the noise correlation functions $R_{\alpha\beta}[n]$, where

$$R_{\alpha\beta}[n] = \langle \Delta\alpha[n]\Delta\beta[0]\rangle. \tag{3}$$

The Greek letters $\{\alpha, \beta\}$ stand for any one of $\{e, t\}$. The noise spectral density function $\Phi_{\alpha\beta}(\Omega)$ is the discrete-time Fourier transform of the noise correlation function,[25]

$$\Phi_{\alpha\beta}(\Omega) = \sum_{n=-\infty}^{\infty} R_{\alpha\beta}[n]\exp(-j\Omega T n). \tag{4}$$

From the definition of the noise spectral density functions in Eq. (4), it is obvious that they are periodic in frequency with a period equal to the pulse repetition frequency $2\pi/T$, i.e., $\Phi_{\alpha\beta}(\Omega + 2\pi/T) = \Phi_{\alpha\beta}(\Omega)$. If the noise spectral density functions $\Phi_{\alpha\beta}(\Omega)$ are known, the noise correlation functions can be found by the inverse Fourier transformation,[25]

$$R_{\alpha\beta}[n] = T\int_{-\pi/T}^{\pi/T} \frac{\mathrm{d}\Omega}{2\pi} \Phi_{\alpha\beta}(\Omega)\exp(j\Omega T n). \tag{5}$$

It follows from Eq. (5) that the mean square values, ${\sigma_e}^2$ and ${\sigma_t}^2$, for the pulse energy and the timing fluctuations, respectively, are

$${\sigma_e}^2 = R_{ee}[0] = T\int_{-\pi/T}^{\pi/T} \frac{\mathrm{d}\Omega}{2\pi} \Phi_{ee}(\Omega), \tag{6}$$

$${\sigma_t}^2 = R_{tt}[0] = T\int_{-\pi/T}^{\pi/T} \frac{\mathrm{d}\Omega}{2\pi} \Phi_{tt}(\Omega). \tag{7}$$

Note that the full integration bandwidth in Eqs. (6) and (7) equals the pulse repetition frequency $2\pi/T$.

The pulse noise spectral density functions can be determined by measurement of the spectral density of the photodetector current noise.[18] In Appendix A it is shown that the spectral density $S_{II}(\Omega)$ of the photodetector current noise can be expressed in terms of the pulse noise spectral density functions defined above in the following way:

$$\begin{aligned} S_{II}(\Omega) \approx (\eta_r P)^2 \bigg\{ & \frac{1}{P^2 T}\Phi_{ee}(\Omega) + \Omega^2 T \Phi_{tt}(\Omega) \\ & + \frac{j\Omega}{P}[\Phi_{et}(\Omega) - \Phi_{et}{}^*(\Omega)] \bigg\}, \end{aligned} \tag{8}$$

where η_r is the responsivity (units amperes/watts) of the photodetector and P is the average power E_p/T. The residual phase noise technique can be used to obtain the noise spectral density for the timing noise in the pulses relative to the phase noise of the RF oscillator.[4–7,9,26] Details are given in Appendix A. Measurement of the pulse noise spectral density functions $\Phi_{\alpha\beta}(\Omega)$ provides a complete description of the energy and the timing noise in optical pulses irrespective of whether the optical pulses are from a harmonically mode-locked laser or from a fundamentally mode-locked laser. In the following sections we show how the noise correlation functions and the noise spectral density functions of harmonically mode-locked lasers are different from those of fundamentally mode-locked lasers. In a fundamentally mode-locked laser the pulse repetition rate $1/T$ equals $1/T_R$, where T_R is the cavity round-trip time and the cavity round-trip frequency is $\Omega_R = 2\pi/T_R$. In a laser mode locked at the Nth harmonic, the pulse repetition rate is $1/T_N = N/T_R$, and the cavity round-trip frequency is $\Omega_N = N\Omega_R$. This notation will be used in the rest of this paper.

3. SUPERMODES AND NOISE CORRELATIONS

The essential difference between the noise spectral density functions of harmonically and fundamentally locked lasers is that the noise spectral density functions of harmonically mode-locked lasers have noise peaks at multiples of the cavity round-trip frequency Ω_R in addition to the noise peaks at multiples of the pulse repetition frequency Ω_N $(= N\Omega_R)$. These additional noise peaks have been called the supermode noise peaks in the literature.[4–6,12,19–21] A harmonically mode-locked laser can have many different supermodes lasing at the same time.[3] Each supermode consists of a set of laser cavity modes that are spaced apart in frequency by $N\Omega_R$. The center frequency of a supermode differs from the center frequencies of the other supermodes by integral multiples of Ω_R. In actively harmonically mode-locked lasers, the phase of all the cavity modes belonging to the same supermode are locked by the active modulation. The supermode noise peaks in the noise spectral density functions are usually related to the beating between different supermodes.[12,19–21,27] Our aim in this paper is to show that the supermode noise peaks are directly related to the nature of the correlations in the noise in different pulses inside the laser cavity.

The dominant sources of the pulse energy and timing noise in mode-locked lasers are spontaneous emission and vacuum fluctuations.[22–24,28] The spontaneous emission noise, as well as the vacuum fluctuation noise, that goes into a pulse in a harmonically mode-locked laser is independent of the spontaneous emission noise in other pulses, and therefore the noise in different pulses is mostly uncorrelated. Correlations in the noise in different pulses can arise in various ways, some of which are described below.

Gain Dynamics. The gain recovery times in fiber and semiconductor mode-locked lasers can be much longer than the pulse repetition times. Since all the pulses interact with the same gain medium, the noise in different pulses can become correlated. Pulse energy fluctuations that are positively correlated in all the pulses are damped effectively by the gain medium, since the slow gain medium responds to the average energy of the all the pulses. On the other hand, pulse energy fluctuations that are negatively correlated in the pulses and do not affect the average power are not damped by the slow response of the gain medium. As a result, negatively correlated pulse energy fluctuations can grow, causing instability and pulse dropouts. Dynamic nonlinearities, such as the Kerr effect in fiber lasers, can stabilize harmonically mode-locked operation.[8,29–31] When these dynamic nonlinearities are small, the pulse energy fluctuations can become negatively correlated. Numerical simulations of harmonically mode-locked semiconductor lasers in Ref. 32 indicate negatively correlated energy fluctuations in the pulses inside the laser cavity.

Optical Fabry–Perot Filters. Experimental results for harmonically mode-locked lasers with optical Fabry–Perot filters placed inside the laser cavities have been reported in Refs. 20, 33, and 34. If a high-*Q* Fabry–Perot filter with a free spectral rangle equal to the pulse repetition frequency Ω_N ($= N\Omega_R$) is placed inside a harmonically mode-locked laser cavity, then some fraction of the noise in each optical pulse will be transferred to the subsequent pulse because of the Fabry–Perot cavity. As a result, the noise in all the pulses in the laser cavity will become positively correlated. If the free spectral rangle of the Fabry–Perot cavity is $m\Omega_R$, where m is some nonzero integer less than N such that N is a multiple of m, then some fraction of the noise in each pulse is injected into the mth subsequent pulse. Consequently, the noise in every mth pulse inside the laser cavity will become positively correlated.

Composite Cavity Harmonically Mode-Locked Lasers. Recently experimental results were reported in Refs. 21 and 27 for the noise spectral densities in composite-cavity harmonically mode-locked fiber lasers. In these lasers an all-fiber Mach–Zehnder interferometer with unbalanced arms is inserted into the laser cavity. If the difference in the time taken by the optical pulses to traverse the two arms of the interferometer equals mT_N, where m is some nonzero integer less than N such that N is a multiple of m, then some fraction of the noise in each pulse is injected into the mth subsequent pulse, and therefore the noise in every mth pulse will become positively correlated.

RF Oscillator Noise. In actively harmonically mode-locked lasers all the optical pulses in the laser cavity are driven by the same active modulator. The amplitude and phase noise in the RF oscillator is typically at low frequencies. Therefore the noise in the pulses coming from the amplitude or phase noise of the external RF oscillator is expected to be positively correlated in all the pulses in the laser cavity. We consider this case in greater detail below.

The list above is not meant to be exhaustive but is intended only to give the readers some concrete examples. Below we discuss in detail the noise correlation functions and the noise spectral density functions of fundamentally and harmonically mode-locked lasers.

4. CASE I: FUNDAMENTALLY MODE-LOCKED LASERS

In a fundamentally mode-locked laser the noise spectral density functions $\Phi_{\alpha\beta}(\Omega)$ have identical noise peaks at multiples of the pulse repetition frequency Ω_R.[18,22] The width of the noise peaks depends on how fast the noise correlation function $R_{\alpha\beta}[n]$ decays with $|n|$. As an example, we consider below the pulse timing noise in an actively mode-locked semiconductor laser in the absence of dispersion and active phase modulation. In the sections that follow, we modify the example discussed below for harmonically mode-locked lasers.

A. Example: Timing Noise in a Fundamentally Mode-Locked Semiconductor Laser

A finite-difference equation for the pulse timing noise $\Delta t[n]$ at any location inside the cavity can be derived from the models presented in Refs. 22–24 (see Appendix B for details),

$$\Delta t[n + 1] - \Delta t[n] = -\gamma T_R \Delta t[n] + F[n]. \tag{9}$$

Equation (9) expresses the fact that the pulse timing noise decreases after every pass through the active modulator. It is assumed that $\gamma T_R \ll 1$ and that the pulse timing noise does not change significantly in one round trip. We have chosen to model the pulse timing noise by a discrete-time finite-difference equation instead of a continuous-time differential equation as in Refs. 22 and 23. It will become clear in the sections that follow that finite-difference equations are much more appropriate for describing the correlations in the noise in different pulses inside the laser cavity in harmonically mode-locked lasers. $F[n]$ in Eq. (9) represents the contribution from spontaneous emission and vacuum fluctuations added to the pulse timing noise in each round trip. $F[n]$ has the correlation function

$$\langle F[n]F[m]\rangle = D\,\delta_{n,m}\,. \tag{10}$$

Expressions for D and γ are given in Appendix B. The timing noise correlation function $R_{tt}[n]$ for the output pulses can be obtained directly from Eqs. (9) and (10), and after a little algebra we obtain

$$R_{tt}[n] = \frac{D}{2\gamma T_R}(1 - \gamma T_R)^{|n|} \approx \frac{D}{2\gamma T_R}\exp(-\gamma T_R|n|), \qquad \text{since} \quad \gamma T_R \ll 1. \tag{11}$$

The mean square timing jitter ${\sigma_t}^2$ is

$${\sigma_t}^2 = R_{tt}[0] = D/2\gamma T_R\,. \tag{12}$$

The correlation function $R_{tt}[n]$ is shown in Fig. 1 for $T_R = 1.0$ ns, $\gamma T_R = 0.01\pi$, and $\sigma_t = 100$ fs. The timing noise spectral density can be obtained from the correlation function by use of the Fourier transform relation in Eq. (4),

$$\Phi_{tt}(\Omega) = {\sigma_t}^2 \frac{2\gamma T_R}{1 + (1 - \gamma T_R)^2 - 2(1 - \gamma T_R)\cos(\Omega T_R)} \tag{13}$$

$$\approx \frac{{\sigma_t}^2}{T_R}\sum_{n=-\infty}^{\infty}\frac{2\gamma}{(\Omega - n\Omega_R)^2 + \gamma^2}, \qquad \text{since} \quad \gamma T_R \ll 1. \tag{14}$$

As expected, the timing noise spectral density has identical noise peaks centered at multiples of the pulse repetition frequency Ω_R. The timing noise spectral density is shown in Fig. 2 for the correlation function shown in Fig. 1, and the values of the parameters are assumed to be the same as those in Fig. 1.

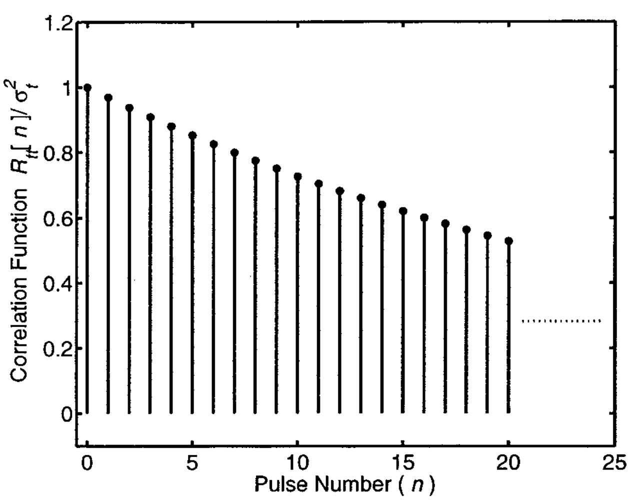

Fig. 1. Timing noise correlation function $R_{tt}[n]$ (normalized to the RMS timing jitter) plotted for the output pulses from a fundamentally mode-locked laser. T_R is assumed to be 1 ns; γT_R is 0.01π. The RMS timing jitter is assumed to be 100 fs.

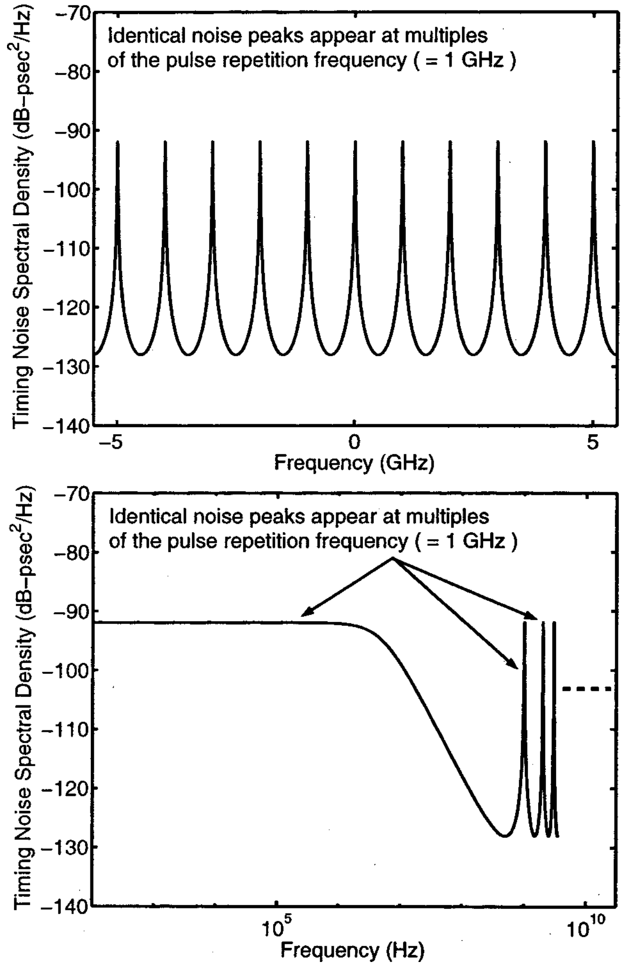

Fig. 2. Timing noise spectral density $T_R\Phi_{tt}(\Omega)$ (note the multiplication by T_R to conform to the units used in the literature) plotted for a fundamentally mode-locked laser on a linear frequency scale and on a log frequency scale. The timing noise spectral density shown corresponds to the timing noise correlation function in Fig. 1. The cavity round-trip time T_R is 1.0 ns; γT_R equals 0.01π. The RMS timing jitter is assumed to be 100 fs. The spectral density has identical noise peaks at multiples of the pulse repetition frequency Ω_R.

B. Integration Bandwidth for the Mean Square Fluctuations

Equations (6) and (7) show that the mean square values for the pulse energy and timing fluctuations can be obtained from the noise spectral density functions $\Phi_{ee}(\Omega)$ and $\Phi_{tt}(\Omega)$, respectively, by integration over a bandwidth equal to Ω_R. Relation (8) shows that the mean square value for the pulse energy fluctuations can be determined from the experimentally measured photodetector current noise spectral density $S_{II}(\Omega)$ by integration around $\Omega = 0$, where the contribution to $S_{II}(\Omega)$ from the pulse timing noise is negligible,[18]

$${\sigma_e}^2[\text{measured}] = \int_{-\Omega_R/2}^{\Omega_R/2}\frac{d\Omega}{2\pi}{T_R}^2\frac{S_{II}(\Omega)}{{\eta_r}^2}. \tag{15}$$

The mean square value for the pulse timing fluctuations can be found by integration of $S_{II}(\Omega)$ near a sufficiently

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large harmonic m of the pulse repetition frequency, where the noise contribution from the pulse energy fluctuations is expected to be small,[18]

$$\sigma_t{}^2[\text{measured}] = \int_{m\Omega_R-\Omega_R/2}^{m\Omega_R+\Omega_R/2} \frac{\mathrm{d}\Omega}{2\pi}\, \frac{S_{II}(\Omega)}{(\Omega\,\eta_r P)^2}. \tag{16}$$

5. CASE II: HARMONICALLY MODE-LOCKED LASERS—UNCORRELATED NOISE

In this section we show that the supermode noise peaks appear in the noise spectral density functions of harmonically mode-locked lasers when the noise in different pulses inside the laser cavity is uncorrelated. The correlations in the noise in the pulses coming out of the laser cavity at time scales shorter than the cavity round-trip time are directly related to the noise correlations in the pulses in the laser cavity. If the noise in the pulses inside the laser cavity is uncorrelated, the correlation function $R_{\alpha\beta}[n]$ for the noise in the output pulses is zero unless the index n is some multiple of the harmonic number N. This is because every Nth output pulse is generated by the same pulse inside the laser cavity after one complete round trip. This observation, without any additional assumptions, leads directly to the supermode noise peaks in the pulse noise spectral density functions, as we show below. From Eq. (4), the noise spectral density becomes

$$\begin{aligned}\Phi_{\alpha\beta}(\Omega) &= \sum_{n=-\infty}^{\infty} R_{\alpha\beta}[n]\exp(-j\Omega T_N n) \\ &= \sum_{k=-\infty}^{\infty} R_{\alpha\beta}[Nk]\exp(-j\Omega T_R k).\end{aligned} \tag{17}$$

Recall that the noise spectral density functions are by definition periodic in frequency with a period equal to the pulse repetition frequency (which is Ω_N in the present case). However, Eq. (17) shows that when the noise is uncorrelated in the pulses inside the laser cavity the noise spectral density functions $\Phi_{\alpha\beta}(\Omega)$ are periodic in frequency with a period equal to the cavity round-trip frequency Ω_R. This implies that if the noise spectral density functions have noise peaks located at integral multiples of the pulse repetition frequency Ω_N, then between any two such noise peaks there must be $(N-1)$ identical noise peaks located at integral multiples of the cavity round-trip frequency Ω_R. These additional noise peaks are the supermode noise peaks. The experimental results published recently in Ref. 35 confirmed that *identical* noise peaks appear in the timing noise spectral density at multiples of the cavity round-trip frequency Ω_R when the noise in different pulses inside the laser cavity is uncorrelated. Below we show this explicitly for the pulse timing noise spectral density in a harmonically mode-locked laser.

A. Example: Timing Noise in a Harmonically Mode-Locked Semiconductor Laser

We consider the pulse timing noise in a semiconductor laser mode-locked at the Nth harmonic with uncorrelated timing noise in different pulses inside the laser cavity. The finite-difference equation for the pulse timing noise is (see Appendix B for details)

$$\Delta t[n+N] - \Delta t[n] = -\gamma_N T_R \Delta t[n] + F_N[n]. \tag{18}$$

Note that $\Delta t[n+N]$ appears on the left-hand side of the above equation. This is because in a harmonically mode-locked laser the nth pulse at any location in the laser cavity becomes the $(n+N)$th pulse at the same location after it goes through one complete round trip. In essence, Eq. (18) is a compact way of writing N separate finite difference equations for the timing noise of N different pulses inside the laser cavity. When $N=1$, Eq. (18) reduces to Eq. (9) for fundamentally mode-locked lasers. The noise source $F_N[n]$ has the correlation function

$$\langle F_N[n]F_N[m]\rangle = D_N\delta_{n,m}. \tag{19}$$

Expressions for D_N and γ_N are given in Appendix B. It is assumed that $\gamma_N T_R \ll 1$ and that the pulse timing noise does not change significantly in one round trip. The timing noise correlation function $R_{tt}[n]$ for the output pulses follows directly from Eqs. (18) and (19),

$$\begin{aligned}R_{tt}[n] &= \frac{D_N}{2\gamma_N T_R}(1-\gamma_N T_N)^{|n|} \\ &\approx \frac{D_N}{2\gamma_N T_R}\exp(-\gamma_N T_N|n|) \\ &\qquad \text{if } n \text{ is an intergral multiple of } N, \\ R_{tt}[n] &= 0 \quad \text{otherwise.}\end{aligned} \tag{20}$$

The mean square timing jitter $\sigma_t{}^2$ is

$$\sigma_t{}^2 = R_{tt}[0] = D_N/2\gamma_N T_R. \tag{21}$$

The timing noise correlation function in Eq. (20) is shown in Fig. 3 for a laser mode locked at the tenth harmonic ($N=10$). In Fig. 3, $T_R = 1.0$ ns, $\gamma T_R = 0.01\pi$, and $\sigma_t = 100$ fs. The timing noise in the output pulses at time scales shorter than the cavity round-trip time T_R is uncorrelated. This is because the timing noise in different pulses in the laser cavity is uncorrelated. The timing

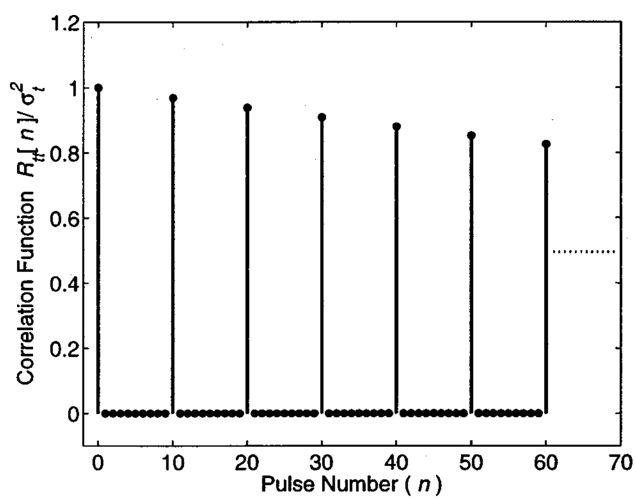

Fig. 3. Timing noise correlation function $R_{tt}[n]$ (normalized to the RMS timing jitter) plotted for the output pulses from a laser mode locked at the tenth harmonic ($N=10$) when the timing noise in different pulses inside the laser cavity is completely uncorrelated. The timing noise in every tenth pulse in the output is correlated. T_R is assumed to be 1 ns; γT_R is assumed to be 0.01π. The RMS timing jitter is assumed to be 100 fs.

noise spectral density can be obtained from the correlation function in Eq. (20) by use of the Fourier transform relation in Eq. (4),

$$\Phi_{tt}(\Omega) = \sigma_t^2 \frac{2\gamma_R T_R}{1 + (1 - \gamma_N T_R)^2 - 2(1 - \gamma_N T_R)\cos(\Omega T_R)} \quad (22)$$

$$\approx \frac{\sigma_t^2}{NT_N} \sum_{n=-\infty}^{\infty} \frac{2\gamma_N}{(\Omega - n\Omega_R)^2 + \gamma_N^2}, \quad \text{since} \quad \gamma_N T_R \ll 1. \quad (23)$$

Relations (22) and (23) show that the timing noise spectral density has identical noise peaks at multiples of the cavity round-trip frequency Ω_R. The noise peaks other than the ones at multiples of the pulse repetition frequency Ω_N are the supermode noise peaks. The supermode noise peaks in the timing noise spectral density are a direct consequence of the timing noise's being uncorrelated in different pulses inside the laser cavity. The timing noise spectral density function in relation (23) is plotted in Fig. 4 for a laser mode locked at the tenth harmonic ($N = 10$). The values assumed for the parameters are the same as those used in generating Fig. 3.

B. Integration Bandwidth for the Mean Square Fluctuations

When the noise in different pulses in the laser cavity is uncorrelated, the pulse noise spectral density functions are periodic in frequency with a period equal to Ω_R, and the mean square values for the pulse energy and timing fluctuations can be obtained by integration of $\Phi_{ee}(\Omega)$ and $\Phi_{ee}(\Omega)$, respectively, over a bandwidth equal to Ω_R instead of the full bandwidth Ω_N:

$$\sigma_e^2 = T_N \int_{-\Omega_N/2}^{\Omega_N/2} \frac{d\Omega}{2\pi} \Phi_{ee}(\Omega) \quad (24)$$

$$= NT_N \int_{-\Omega_R/2}^{\Omega_R/2} \frac{d\Omega}{2\pi} \Phi_{ee}(\Omega), \quad (25)$$

$$\sigma_t^2 = T_N \int_{-\Omega_N/2}^{\Omega_N/2} \frac{d\Omega}{2\pi} \Phi_{tt}(\Omega) \quad (26)$$

$$= NT_N \int_{-\Omega_R/2}^{\Omega_R/2} \frac{d\Omega}{2\pi} \Phi_{tt}(\Omega). \quad (27)$$

The integration bandwidth can therefore be reduced to the cavity round-trip frequency Ω_R, provided the result is multiplied by N.

The mean square value for the pulse energy fluctuations can be determined from the experimentally measured photodetector current noise spectral density $S_{II}(\Omega)$ by integrating near $\Omega = 0$, where the contribution from the pulse timing fluctuations is expected to be small:

$$\sigma_e^2[\text{measured}] = \int_{-\Omega_N/2}^{\Omega_N/2} \frac{d\Omega}{2\pi} T_N^2 \frac{S_{II}(\Omega)}{\eta_r^2} \quad (28)$$

$$= N \int_{-\Omega_R/2}^{\Omega_R/2} \frac{d\Omega}{2\pi} T_N^2 \frac{S_{II}(\Omega)}{\eta_r^2}. \quad (29)$$

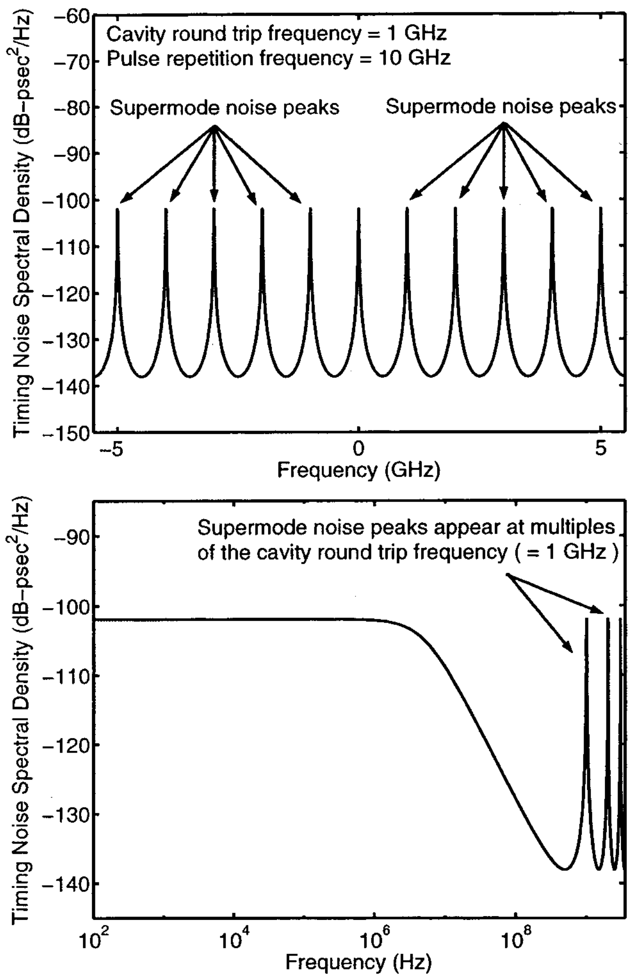

Fig. 4. Pulse timing noise spectral density $T_N\Phi_{tt}(\Omega)$ (note the multiplication by T_N to conform to the units used in the literature) plotted for a laser mode locked at the tenth harmonic ($N = 10$) on linear and log frequency scales. The timing noise spectral density shown in the figure corresponds to the timing noise correlation function in Fig. 3. T_R is 1.0 ns; T_N is 0.1 ns; $\gamma_N T_R$ is assumed to be 0.01π. The RMS timing jitter is assumed to be 100 fs. The timing noise in different pulses inside the laser cavity is assumed to be completely uncorrelated and, consequently, the periodicity of the noise spectral density is reduced from the pulse repetition frequency (10 GHz) to the cavity round-trip frequency (1 GHz). Supermode noise peaks appear at multiples of the cavity round-trip frequency. All the noise peaks are identical.

Assuming that the pulse energy fluctuations make small contributions to $S_{II}(\Omega)$ near a large harmonic number m of the pulse repetition frequency, the mean square value for the pulse timing fluctuations can be determined as follows:

$$\sigma_t^2[\text{measured}] = \int_{(m-1/2)\Omega_N}^{(m+1/2)\Omega_N} \frac{d\Omega}{2\pi} \frac{S_{II}(\Omega)}{(\Omega\eta_r P)^2} \quad (30)$$

$$= N \int_{m\Omega_N-\Omega_R/2}^{m\Omega_N+\Omega_R/2} \frac{d\Omega}{2\pi} \frac{S_{II}(\Omega)}{(\Omega\eta_r P)^2}. \quad (31)$$

The pulse timing noise contribution to the supermode noise peaks in the photodetector current noise spectral density was experimentally observed in Ref. 5. With the exception of the experimental work presented in Refs. 5

and 21, the supermode noise peaks have been largely ignored in the literature. To the best of our knowledge, all the experimental results presented in the literature for the mean square energy and timing noise of pulses in harmonically mode-locked lasers have left out the contribution to the mean square fluctuations from the supermode noise peaks.[4–6,10,13] When the noise is uncorrelated in different pulses inside the laser cavity, Eqs. (29) and (31) show that ignoring the contribution from the supermode noise peaks gives mean square fluctuations that are less than the correct values by a factor of N. The procedure for determining the mean square pulse fluctuations by integrating only over a bandwidth equal to the cavity round-trip frequency Ω_R and multiplying the result by N is justified only if all the noise peaks are identical. The noise peaks are not identical when the noise in different pulses in the laser cavity is correlated.

6. CASE III: HARMONICALLY MODE-LOCKED LASERS—CORRELATED NOISE

In Section 5 it was shown that the noise spectral density functions can have N noise peaks in a bandwidth equal to the pulse repetition frequency Ω_N. When the noise in different pulses in the cavity is uncorrelated, all N noise peaks are identical and have the same spectral weight. Here, we show how the spectral weights of the different noise peaks are modified when the noise in different pulses is correlated. As mentioned above, the noise correlation function $R_{\alpha\beta}[n]$ for the output pulses at time scales shorter than the cavity round-trip time (i.e., for $|n| < N$) is a good measure of the noise correlation in pulses inside the laser cavity. For $|n| < N$, $R_{\alpha\beta}[n]$ given by Eq. (5) can be approximated as

$$R_{\alpha\beta}[n] = T_N \int_{-\Omega_N/2}^{\Omega_N/2} \frac{d\Omega}{2\pi} \Phi_{\alpha\beta}(\Omega) \exp(j\Omega T_N n) \tag{32}$$

$$\approx \sum_{p=0}^{N-1} \exp\left(j2\pi \frac{p}{N} n\right) T_N \int_{(p-1/2)\Omega_R}^{(p+1/2)\Omega_R} \frac{d\Omega}{2\pi} \Phi_{\alpha\beta}(\Omega)$$

$$= R_{\alpha\beta}[0] \sum_{p=0}^{N-1} W_{\alpha\beta}{}^{p} \exp\left(j2\pi \frac{p}{N} n\right)$$

$$= R_{\alpha\beta}[0] C_{\alpha\beta}[n], \tag{33}$$

where $W_{\alpha\beta}{}^{p}$ is the spectral weight of the pth noise peak,

$$W_{\alpha\beta}{}^{p} = \frac{\displaystyle\int_{(p-1/2)\Omega_R}^{(p+1/2)\Omega_R} \frac{d\Omega}{2\pi} \Phi_{\alpha\beta}(\Omega)}{\displaystyle\int_{-\Omega_N/2}^{\Omega_N/2} \frac{d\Omega}{2\pi} \Phi_{\alpha\beta}(\Omega)}, \tag{34}$$

and

$$C_{\alpha\beta}[n] = \frac{R_{\alpha\beta}[n]}{R_{\alpha\beta}[0]}$$

$$= \sum_{p=0}^{N-1} W_{\alpha\beta}{}^{p} \exp\left(j2\pi \frac{p}{N} n\right) \qquad \text{for} \quad |n| < N. \tag{35}$$

In deriving Eq. (33), we assumed that the complex exponential function in relation (32) is slowly varying with frequency and can be set equal to its value at the center frequency of each noise peak. This approximation is valid for small values of the index n, provided that the width of each noise peak is much smaller than the separation Ω_R of the noise peaks. For most fiber and semiconductor mode-locked lasers this approximation is well justified.[4–7,9,10,13,22,24,28] The function $C_{\alpha\beta}[n]$ defined above describes the correlations in the noise in different pulses inside the laser cavity. $C_{\alpha\beta}[n]$ satisfies the relations $-1 \leqslant C_{\alpha\beta}[n] \leqslant 1$ and $C_{\alpha\beta}[n + N] = C_{\alpha\beta}[n]$. The value of $C_{\alpha\beta}[n]$ gives the correlation, on a scale from -1 to 1, in the noise of any two pulses in the cavity that are separated by $n - 1$ other pulses. Since the pulse noise spectral densities can be determined experimentally by measurement of the spectral density of the photodetector current noise, the correlations in the noise in different pulses inside the laser cavity can be determined by use of the result in Eq. (35). The result in Eq. (35) shows that $C_{\alpha\beta}[n]$ is equal to the Fourier transform of the spectral weight of all N noise peaks in $\Phi_{\alpha\beta}(\Omega)$ in a bandwidth equal to the pulse repetition frequency Ω_N. The following results can be obtained from this Fourier transform relationship:

Noise peaks in the spectral density near multiples of Ω_N will have larger spectral weights if the noise in all the pulses in the laser cavity is positively correlated. As a special case, suppose that the spectral weights of all the supermode noise peaks in $\Phi_{\alpha\beta}(\Omega)$ are negligible and only the noise peaks at multiples of the pulse repetition frequency Ω_N have the entire spectral weight. It follows from Eq. (35) that $C_{\alpha\beta}[n] = 1$ for all values of the index n, and the noise is *completely* positively correlated in all the pulses in the laser cavity.

Noise peaks in the spectral density near odd multiples of $\Omega_N/2$ will have larger spectral weights if the noise in the neighboring pulses inside the laser cavity is negatively correlated. For example, suppose that N is even and the supermode noise peaks at odd multiples of $\Omega_N/2$ have the entire spectral weight. In this case, $C_{\alpha\beta}[n] = (-1)^n$ and the noise is *completely* negatively correlated in the neighboring pulses.

If all the noise peaks in the spectral density have the same spectral weight, then $C_{\alpha\beta}[n] = 1$ for $n = 0$, and $C_{\alpha\beta}[n] = 0$ for $1 \leqslant |n| < N$, and the noise is uncorrelated in different pulses inside the laser cavity. This case was also discussed in Section 5.

In Ref. 21 experimental results were recently reported for the pulse noise spectral densities of composite-cavity harmonically mode-locked fiber lasers. In these lasers a fraction of each pulse can be injected into the mth subsequent pulse by an all-fiber Mach–Zehnder interferometer with unbalanced arms placed inside the laser cavity. This is expected to positively correlate the noise in every mth pulse in the cavity. The noise in any two pulses that are not separated by $m - 1$ other pulses is expected to remain uncorrelated. When the spectral weights of the noise peaks given in Fig. 7 of Ref. 21 for different values of m are used in Eq. (35), the resulting correlation functions $C_{\alpha\beta}[n]$ confirm these expected noise correlations.

A. Example: Timing Noise in a Harmonically Mode-Locked Semiconductor Laser

In the Section 5 a model for the timing noise in harmonically mode-locked semiconductor laser was presented for the case in which the timing noise in the pulses inside the laser cavity was uncorrelated. Here we include the contribution to the pulse timing noise from the phase noise in the RF oscillator. It will be shown that since the noise contribution from the RF oscillator is completely positively correlated in all the pulses inside the laser cavity, this noise contribution shows up in the timing noise spectral density only in the noise peaks located at multiples of the pulse repetition frequency Ω_N and not in the supermode noise peaks. On the other hand, the contribution to the timing noise from spontaneous emission and vacuum fluctuations is uncorrelated in different pulses inside the laser cavity and shows up equally in all the noise peaks (including the supermode noise peaks). We assume that the electrical signal from the RF oscillator is proportional to $\cos\{\Omega_N[t - \Delta J(t)]\}$, where $\Delta J(t)$ is the timing noise in the RF oscillator. $\Delta J(t)$ is assumed to have the correlation function

$$\langle J(t)J(t')\rangle = {\sigma_{\mathrm{RF}}}^2 \exp(-\kappa|t - t'|), \tag{36}$$

which implies that the mean square timing noise of the RF oscillator equals ${\sigma_{\mathrm{RF}}}^2$ and the bandwidth of the RF oscillator noise equals $\kappa/2\pi$ Hz. Typically, the phase noise in RF oscillators is mostly at low frequencies[7]; therefore $\kappa \ll \Omega_R$, and the contribution to the pulse timing noise from the phase noise of the RF oscillator is expected to be positively correlated in all the pulses inside the laser cavity. We define the discrete noise variable $\Delta J[n]$ as equal to $\Delta J(t = nT_N)$.

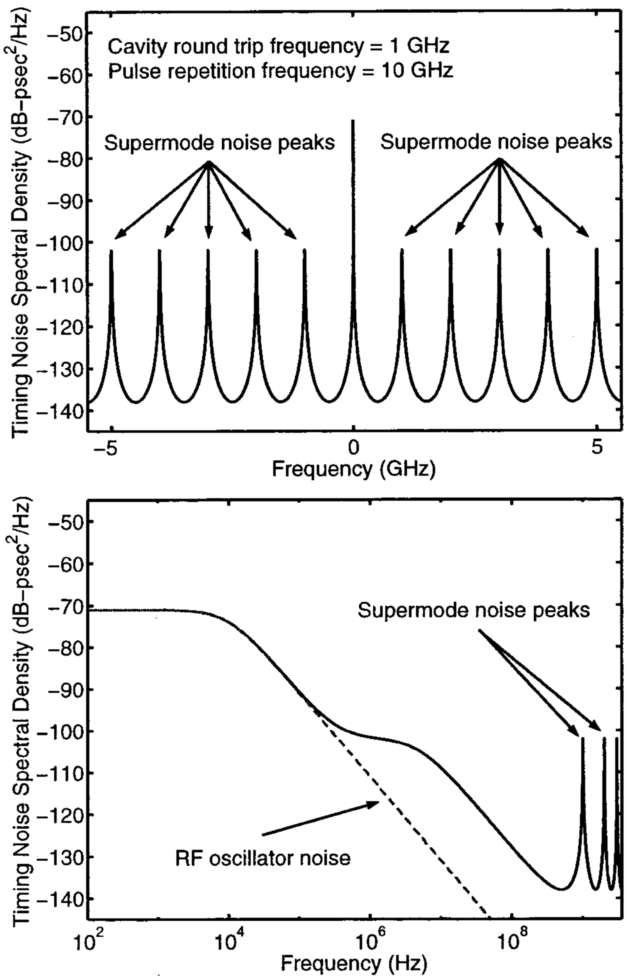

Fig. 5. Pulse timing noise spectral density $T_N\Phi_{tt}(\Omega)$ (solid curve) plotted for a laser mode locked at the tenth harmonic ($N = 10$) in the presence of timing (or phase) noise in the RF oscillator (dashed curve) on linear and log frequency scales. T_R is 1.0 ns; T_N is 0.1 ns; $\gamma_N T_R$ and κT_R are assumed to be 0.01π and $2\pi 10^{-5}$, respectively. The RMS timing jitter in the RF oscillator is assumed to be 50 fs. The RMS timing jitter contribution from spontaneous emission and vacuum fluctuations is assumed to be 100 fs. The increased noise in the noise peak at $\Omega = 0$ is due to the phase noise of the RF oscillator. The figure shows that the noise contribution form the RF oscillator does not appear in any of the supermode noise peaks. All the supermode noise peaks are identical.

To study the correlations in the noise in different pulses, we must include the noise in all the pulses in the model. The finite difference equations for the timing noise introduced earlier are most suitable for this purpose and allow the noise in all the pulses to be taken into account in a relatively straightforward way. In the presence of phase noise in the oscillator, the finite difference equation for the pulse timing noise becomes

$$\begin{aligned}\Delta t[n + N] - \Delta t[n] = -\gamma_N T_R \Delta t[n] + \gamma_N T_R \Delta J[n] \\ + F_N[n].\end{aligned} \tag{37}$$

Equation (37) is identical to Eq. (18) except for the term with $\Delta J[n]$. The form of this new term follows from the fact that the pulse cannot be affected by the modulator if $\Delta t[n]$ equals $\Delta J[n]$. It is difficult to solve Eq. (37) directly, but it can easily be solved in the frequency domain by use of the discrete-time Fourier transform,[25] and we obtain the following expression for the pulse timing noise spectral density:

$$\begin{aligned}\Phi_{tt}(\Omega) \\ = \frac{1}{1 + (1 - \gamma_N T_R)^2 - 2(1 - \gamma_N T_R)\cos(\Omega T_R)} \\ \times \left\{ D_N + \frac{{\sigma_{\mathrm{RF}}}^2 {\gamma_N}^2 {T_R}^2 [1 - \exp(-2\kappa T_N)]}{1 + \exp(-2\kappa T_N) - 2\exp(-\kappa T_N)\cos(\Omega T_N)} \right\}\end{aligned} \tag{38}$$

$$\begin{aligned}\approx \frac{1}{T_N} \sum_{n=-\infty}^{\infty} \frac{2\gamma_N}{(\Omega - n\Omega_R)^2 + {\gamma_N}^2} \left[\frac{1}{N} \left(\frac{D_N}{2\gamma_N T_R} \right) \right. \\ \left. + \frac{\gamma_N {\sigma_{\mathrm{RF}}}^2}{2} \sum_{p=-\infty}^{\infty} \frac{2\kappa}{(\Omega - p\Omega_N)^2 + \kappa^2} \right].\end{aligned} \tag{39}$$

The first denominator in Eq. (39) represents the timing noise contribution from spontaneous emission and vacuum fluctuations, and it is identical to the expression given earlier in Eq. (23). Since the timing noise from spontaneous emission and vacuum fluctuations is uncorrelated in different pulses inside the cavity, its contribution to the timing noise spectral density results in identical noise peaks at multiples of the cavity round-trip frequency Ω_R. The second denominator in Eq. (39) is the

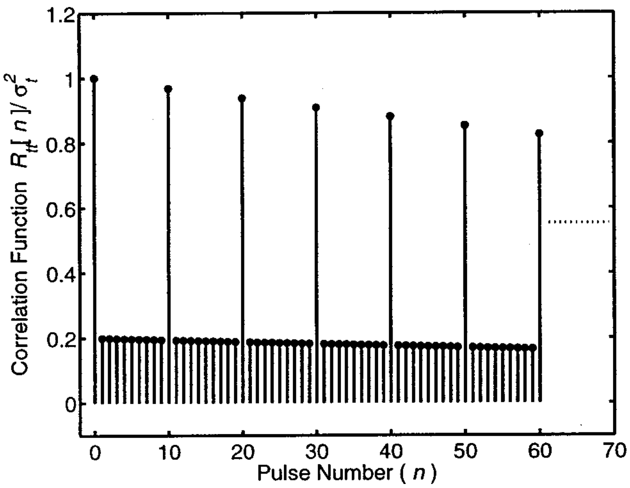

Fig. 6. Timing noise correlation function $R_{tt}[n]$ (normalized to the RMS timing jitter) for the output pulses shown for a laser mode locked at the tenth harmonic ($N = 10$) in the presence of phase noise from the RF oscillator. The correlation function corresponds to the timing noise spectral density shown in Fig. 5. The timing noise in all the pulses inside the laser cavity is positively correlated, and therefore the timing noise in the output pulses is correlated at time scales shorter than the cavity round-trip time.

noise contribution from the phase noise of the RF oscillator. The noise contribution from the oscillator results in noise peaks at multiples of the pulse repetition frequency Ω_N. Since it has been assumed that $\kappa \ll \Omega_R$, the phase noise from the RF oscillator does not contribute to the supermode noise peaks. From the discussion above it follows that the noise contribution from the RF oscillator is completely positively correlated in the pulses inside the laser cavity. This conclusion can be easily tested experimentally by comparing the supermode noise peaks in the timing noise spectral density to the noise peaks at multiples of the pulse repetition frequency. If $\kappa \ll \gamma_N$, Eq. (39) gives the expected result for the mean square timing jitter upon integration,

$$\sigma_t^{\,2} = \frac{D_N}{2\gamma_N T_R} + \sigma_{\mathrm{RF}}^{\,2}. \tag{40}$$

The first term on the right-hand side of the above equation is the mean square timing noise contribution from spontaneous emission and vacuum fluctuations. Figure 5 shows the timing noise spectral density for a laser mode locked at the tenth harmonic ($N = 10$). The parameter values used in generating these figures are as follows: $T_R = 1.0$ ns, $T_N = 0.1$ ns, $\gamma_N T_R = 0.01\pi$, $\kappa T_R = 2\pi 10^{-5}$, and $\sigma_{\mathrm{RF}} = 50$ fs. The rms contribution to the timing noise from spontaneous emission and vacuum fluctuations is assumed to be 100 fs. Figure 5 shows that the contribution to the timing noise from the phase noise of the RF oscillator does not appear in the supermode noise peaks.

The spectral weights $W_{tt}{}^p$, defined in Eq. (34), of the noise peaks can be determined for all N noise peaks in a bandwidth Ω_N for the spectral density given in Eq. (39). Assuming as before that $\kappa \ll \gamma_N$, we obtain

$$W_{tt}{}^p = \frac{(1/N)(D_N/2\gamma_N T_R) + \sigma_{\mathrm{RF}}^{\,2}}{\sigma_t^{\,2}}$$

$$\text{if} \quad p = 0 \quad (\text{noise peaks at multiples of } \Omega_N),$$

$$W_{tt}{}^p = \frac{(1/N)(D_N/2\gamma_N T_R)}{\sigma_t^{\,2}}$$

$$\text{if} \quad 1 \leqslant p \leqslant (N-1)$$

$$(\text{for all the supermode noise peaks}). \tag{41}$$

The correlation function $C_{tt}[n]$, defined in Eq. (35), for the timing noise in different pulses inside the laser cavity can be obtained from the spectral weights of the noise peaks given above, and we get

$$C_{tt}[n] = \begin{cases} 1 & \text{if} \quad n = 0 \\ \sigma_{\mathrm{RF}}^{\,2}/\sigma_t^{\,2} & \text{if} \quad 1 \leqslant |n| \leqslant (N-1) \end{cases}. \tag{42}$$

This implies that the correlation in the timing noise in different pulses inside the laser cavity is $\sigma_{\mathrm{RF}}^{\,2}/\sigma_t^{\,2}$. The timing noise correlation function $R_{tt}[n]$ for the output pulses can be obtained from the spectral density in Eq. (39) by use of the inverse Fourier transform relation in Eq. (5). Figure 6 shows the correlation function $R_{tt}[n]$, calculated numerically, that corresponds to the noise spectral density shown in Fig. 5. The correlation function shows that the timing noise is positively correlated in the output pulses at time scales shorter than the cavity round-trip time T_R. The degree of positive correlation is given by the ratio $\sigma_{\mathrm{RF}}^{\,2}/\sigma_t^{\,2}$, which for the values used in Fig. 6 equals $50^2/\sqrt{100^2 + 50^2} = 0.2$.

B. Integration Bandwidth for the Mean Square Fluctuations

Since the pulse noise spectral density functions $\Phi_{\alpha\beta}(\Omega)$ are not periodic in frequency with a period equal to Ω_R when the noise in the pulses inside the laser cavity is correlated, the mean square value of the fluctuations can be determined only if the noise spectral density functions $\Phi_{\alpha\beta}(\Omega)$ are integrated over the full bandwidth, equal to Ω_N. It follows that the mean square values of the pulse energy and the timing fluctuations can be determined from the experimentally measured photodetector current noise spectral density $S_{II}(\Omega)$ by use of Eqs. (29) and (30). It should be noted that Eqs. (29) and (31) do not hold when the noise in different pulses in the laser cavity is correlated.

7. CONCLUSION

The noise in different pulses inside the laser cavity in harmonically mode-locked lasers is in general correlated, and these noise correlations have been shown to be related to the distribution of the spectral weight among the supermode noise peaks in the pulse noise spectral density functions. Models for the timing noise in harmonically mode-locked semiconductor lasers were presented that demonstrated the relationship between the supermode noise peaks and the correlations in the noise of different pulses. We believe that studying the correlations in the noise of different pulses in harmonically mode-locked lasers will lead to a better understanding of the laser dynamics. Methods to determine the noise correlations from the photo-detector current noise spectral density were also presented. Most experimental results on the noise in harmonically mode-locked lasers that have appeared in the literature have ignored the rich information content in the supermode noise peaks. We hope that the analysis presented in this paper will stimulate greater interest in the nature of the noise correlations in the pulses inside the laser cavity in harmonically mode-locked lasers.

APPENDIX A: MEASUREMENT OF THE PULSE NOISE SPECTRAL DENSITY FUNCTIONS

One way to characterize the noise in mode-locked lasers is by measuring the spectral density of the photodetector current noise.[18] The photodetector current $I(t)$ is related to the intensity $I_p(t)$ of the optical pulse train coming out of the mode-locked laser by the relation

$$\begin{aligned} I(t) &= \eta_r \int_{-\infty}^{\infty} \mathrm{d}t' \chi(t-t') I_p(t') \\ &= \eta_r \int_{-\infty}^{\infty} \mathrm{d}t' \chi(t-t') \sum_n (E_p + \Delta e[n]) f(t' - nT \\ &\quad - \Delta t[n]) \\ &= \eta_r \sum_n (E_p + \Delta e[n]) h(t - nT - \Delta t[n]), \end{aligned}$$

$$h(t) = \int_{-\infty}^{\infty} \mathrm{d}t' \chi(t-t') f(t'), \tag{A1}$$

where η_r is the responsivity of the photodetector (units of amperes/watts); $\chi(t)$ describes the time-dependent response of the photodetector and is normalized such that its integral equals unity. E_p is the energy of a single pulse; $f(t)$ describes the time-independent intensity of a single pulse and is also normalized such that its integral equals unity. The pulse repetition rate is $1/T$; $\Delta e[n]$ and $\Delta t[n]$ are the energy and the timing noise in the nth pulse, respectively. The correlation function $S_{II}(\tau)$ of the photodetector current noise is defined as[36]

$$S_{II}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T_0} \times \int_{-T_0/2}^{T_0/2} \mathrm{d}t \langle [I(t) - \langle I(t) \rangle][I(t+\tau) - \langle I(t+\tau) \rangle] \rangle. \tag{A2}$$

Using Eq. (A1) and defining the average optical power P as E_p/T, we obtain the spectral density $S_{\mathrm{II}}(\Omega)$,

$$S_{II}(\Omega) = |H(\Omega)|^2 (\eta_r P)^2 \left\{ \frac{1}{P^2 T} \Phi_{ee}(\Omega) + \Omega^2 T \Phi_{tt}(\Omega) + \frac{j\Omega}{P} [\Phi_{et}(\Omega) - \Phi_{et}^*(\Omega)] \right\}. \tag{A3}$$

$H(\Omega)$ is the Fourier transform of $h(t)$; for values of Ω smaller than the inverse of the photodetector response time and the inverse of the pulse width, $|H(\Omega)| \approx 1$.

A more reliable technique frequently used to measure the timing noise is mixing the photodetector output current with a signal from the same RF oscillator driving the mode-locked laser.[4–7,9,26] By appropriate adjustment of the phase of the RF signal before mixing it with the photodetector current, the noise contribution from the pulse energy fluctuations can be removed, in principle. In addition, the timing fluctuations in the optical pulse train are measured relative to the timing noise (or the phase noise) of the RF oscillator, and therefore this method is called the residual phase noise method. In this case the signal $M(t)$ at the output of the mixer is

$$M(t) = I(t) a \cos\{\Omega_m [t - \Delta J(t)] + \Phi\}, \tag{A4}$$

where $\Delta J(t)$ is the timing noise of the RF oscillator and Ω_m is the modulation frequency. Noting that for small frequencies $|H(\Omega)| \approx |H(0)| = 1$, and any phase associated with $H(0)$ can be absorbed in the definition of ϕ above, the spectral density $S_{MM}(\Omega)$ of the noise in the signal $M(t)$ is

$$\begin{aligned} S_{MM}(\Omega) &= a^2 (\cos^2 \phi) S_{II}(\Omega) + a^2 (\sin^2 \phi) \\ &\quad \times (\eta_r P)^2 {\Omega_m}^2 T \Phi_{t-Jt-J}(\Omega) \qquad \text{(A5)} \\ &\quad - a^2 (\cos \phi)(\sin \phi) {\eta_r}^2 P \Omega_m [\Phi_{et-J}(\Omega) \\ &\quad + \Phi_{et-J}^*(\Omega)]. \qquad \text{(A6)} \end{aligned}$$

The spectral density function $\Phi_{t-Jt-J}(\Omega)$ of the residual timing noise is defined as

$$\Phi_{t-Jt-J}(\Omega) = \sum_{n=\infty}^{\infty} \langle (\Delta t[n] - \Delta J[n])(\Delta t[0] - \Delta J[0]) \rangle \exp(-j\Omega T n). \tag{A7}$$

In Eq. (A7), $\Delta J[n] = \Delta J(t = nT)$. When the phase ϕ is $\pi/2$, all terms in line (A5) are zero except the one that is proportional to the spectral density of the residual timing fluctuations, and $S_{MM}(\Omega)$ equals

$$S_{MM}(\Omega)|_{\phi=\pi/2} = a^2 (\eta_r P)^2 {\Omega_m}^2 T \Phi_{t-Jt-J}(\Omega). \tag{A8}$$

APPENDIX B: PULSE TIMING NOISE IN FUNDAMENTALLY AND HARMONICALLY MODE-LOCKED SEMICONDUCTOR LASERS

In this paper, models for the pulse timing noise in the absence of dispersion and active phase modulation were presented for fundamentally and harmonically mode-locked semiconductor lasers. In this Appendix we describe the derivation of the discrete-time finite-difference equations for the pulse timing noise in Eqs. (9) and (18) in more detail. The models discussed below are simple, since increasing the complexity of the model does not affect the nature of the conclusions of this paper. We use the time domain pulse perturbation theory developed in Refs. 22 and 23, and modified in Ref. 24 for actively mode-locked semiconductor lasers. The master equation that describes the slow time evolution of the amplitude $\phi(t, T)$ of a pulse in an actively mode-locked semiconductor laser is (see Refs. 22 and 23 for details)

$$T_R \frac{\partial \phi(t, T)}{\partial T} = j\nu + \frac{B}{2}\frac{\partial^2 \phi(t, T)}{\partial t^2} + \frac{1}{2}[G(t) + g_m(t) - L(t)]\phi(t, T) + F_{sp}(t, T) + F_v(t, T), \tag{A9}$$

where the time variable T describes the pulse evolution over time scales longer than the cavity round-trip time T_R. Here ν is the phase shift acquired by the pulse in one round trip. $G(t)$ is the gain. Because of dynamic gain saturation in semiconductor lasers,[24] $G(t)$ is assumed to be a function of the time-dependent pulse intensity. For simplicity, we have ignored dynamic self-phase modulation that accompanies dynamic gain saturation in semiconductor lasers.[24] $L(t)$ is the loss and is also assumed to be a function of the time-dependent pulse intensity because of dynamic loss saturation (i.e., slow saturable absorption). The time-dependent gain caused by the active modulation, $g_m(t)$ is equal to $M[\cos(\Omega_m t) - 1]$, where Ω_m is the modulation frequency. B describes the effect of the finite gain bandwidth (or filter bandwidth). $F_{sp}(t, T)$ and $F_v(t, T)$ represent the noise sources for spontaneous emission and vacuum fluctuations, respectively, and their correlation functions are given in Ref. 23.

1. Fundamentally Mode-Locked Laser

For a fundamentally mode-locked laser, the modulation frequency Ω_m equals the cavity round-trip frequency Ω_R. The steady-state pulse is obtained by solution of the nonlinear master equation in the absence of noise. We assume that the steady-state pulse amplitude is $(n_p/\tau)^{1/2}A(t/\tau)$, where n_p is the number of photons in the pulse and τ is the pulse width. $A(t/\tau)$ is real and is normalized such that $\int_{-\infty}^{\infty}\mathrm{d}x\, A^2(x) = 1$. In the presence of noise, $\phi(t, T)$ is expanded as[22,23]

$$\phi(t, T) \approx \left(\frac{n_p}{\tau}\right)^{1/2} A\left(\frac{t}{\tau}\right) + \left(\frac{n_p}{\tau}\right)^{1/2}\left[\Delta a(T) A\left(\frac{t}{\tau}\right) + j\Delta\theta(T) A\left(\frac{t}{\tau}\right) - \Delta t(T)\frac{\mathrm{d}A}{\mathrm{d}t} - j\Delta\omega(T)(t - t_0) A\left(\frac{t}{\tau}\right)\right], \tag{B1}$$

where $\Delta a(T)$, $\Delta\theta(T)$, $\Delta t(T)$, and $\Delta\omega(T)$ are the pulse amplitude, phase, timing, and frequency fluctuations, respectively. The temporal position t_0 of the steady-state pulse is defined by the equation $\int_{-\infty}^{\infty}\mathrm{d}t(t - t_0)A^2(t/\tau) = 0$. In the presence of dynamic gain or loss saturation, the pulse position t_0 does not coincide with the peak gain in the active modulator.[24] The expansion in relation (B1) ignores the fluctuations that distort the pulse shape. The equation for the pulse amplitude (phase) fluctuations can be obtained by substituting the expansion in (B1) into the master equation and projecting out the pulse amplitude (phase) fluctuations by multiplying both sides of the resulting equation by $A(t/\tau)$, integrating, and keeping the real (imaginary) part. The pulse timing (frequency) fluctuations can be projected out by multiplying both sides by $\mathrm{d}A(t/\tau)/\mathrm{d}t$, integrating, and keeping the real (imaginary) part. This yields

$$\frac{\mathrm{d}\Delta t(T)}{\mathrm{d}T} = -\gamma\Delta t(T) + F(T), \tag{B2}$$

where γ is given by the expression

$$\gamma = \frac{1}{2T_R}\frac{\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\mathrm{d}g_m(t)}{\mathrm{d}t}A(t/\tau)\frac{\mathrm{d}A(t/\tau)}{\mathrm{d}t}}{\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\mathrm{d}A(t/\tau)}{\mathrm{d}t}\frac{\mathrm{d}A(t/\tau)}{\mathrm{d}t}} = \frac{M}{2T_R}{\Omega_R}^2\tau^2\left\{-\frac{\int_{-\infty}^{\infty}\mathrm{d}x\;\; xA(x)A'(x)}{\int_{-\infty}^{\infty}\mathrm{d}x\;\; A'(x)A'(x)}\right\} \tag{B3}$$

$$= \frac{M}{2T_R}{\Omega_R}^2\tau^2\left\{\frac{1}{2\int_{-\infty}^{\infty}\mathrm{d}x\;\; A'(x)A'(x)}\right\}. \tag{B4}$$

Equation (B4) follows from Eq. (B3) because the integral $\int_{\infty}^{\infty}\mathrm{d}x\; xA(x)A'(x)$ equals $-1/2$ for any arbitrary pulse shape. If the pulse shape is approximately Gaussian, the expression in the curly brackets in Eq. (B4) equals unity. The correlation function for the noise source $F(T)$ is given approximately by the expression

$$\langle F(T)F(T') = \frac{\tau^2}{2n_p T_R}[(2n_{sp} - 1)\langle G\rangle + \langle L\rangle] \times \left\{\frac{1}{2\int_{-\infty}^{\infty}\mathrm{d}x\;\; A'(x)A'(x)}\right\}\delta(T - T'), \tag{B5}$$

where $\langle G\rangle$ is the total pulse round-trip gain in the steady state, $\langle L\rangle$ is the pulse round-trip loss, and n_{sp} is the spontaneous emission factor, which takes into account the incomplete inversion of the gain medium.[37] In the steady state, since the round-trip gain $\langle G\rangle$ equals the round-trip loss $\langle L\rangle$, Eq. (B5) can be simplified:

$$\langle F(T)F(T')\rangle = \left(\frac{n_{sp}}{n_p}\right)\frac{\langle L\rangle}{T_R}\tau^2 \times \left\{\frac{1}{2\int_{-\infty}^{\infty}\mathrm{d}x\;\; A'(x)A'(x)}\right\}\delta(T - T'). \tag{B6}$$

Assuming that $\gamma T_R \ll 1$, Eq. (B2) can be discretized:

$$\Delta t[n + 1] - \Delta t[n] = -\gamma T_R \Delta t[n] + F[n]. \tag{B7}$$

The discrete-time noise variable $\Delta t[n]$ is the timing noise in the pulse after the nth round trip. The noise source $F[n]$ represents the total timing noise added to the pulse in one round trip,

$$F[n] = \oint \mathrm{d}TF(T), \tag{B8}$$

and has the correlation function

$$\langle F[n]F[m]\rangle = D\,\delta_{n,m}\,, \tag{B9}$$

where

$$D = \left(\frac{n_{\mathrm{sp}}}{n_p}\right)\langle L\rangle\tau^2\left\{\frac{1}{2\int_{-\infty}^{\infty}\mathrm{d}x\quad A'(x)A'(x)}\right\}. \tag{B10}$$

The expression for the mean square timing jitter in a fundamentally mode-locked laser was given in Eq. (12),

$$\sigma_t^{\,2} = \langle\Delta t^2[n]\rangle = D/2\gamma T_R\,. \tag{B11}$$

Using the values of γ and D given above, we obtain

$$\sigma_t^{\,2} = \left(\frac{n_{\mathrm{sp}}}{n_p}\right)\langle L\rangle\frac{1}{M\Omega_R^{\,2}}. \tag{B12}$$

Note that the pulse round-trip loss $\langle L\rangle$ can be related to the photon lifetime τ_p in the laser cavity, $\langle L\rangle = T_R/\tau_p$. The analysis for the timing noise presented here assumed dynamic gain or loss saturation. The mean square timing jitter is found to be independent of the pulse shape and the pulse width. This is not true for the timing noise spectral density, which depends on the pulse shape and the pulse width through γ [see relations (14) and (B4)]. The model for the pulse timing noise presented here is not applicable when the pulse is chirped and its amplitude is described by a complex function. The pulse can acquire a chirp in the presence of dispersion, active phase modulation, or dynamic self-phase modulation.

2. Harmonically Mode-Locked Laser

A laser mode-locked at the Nth harmonic has N pulses propagating inside the laser cavity. Some additional assumptions are required before we can use the model presented above for harmonically mode-locked lasers. In the noiseless steady state, all the pulses are assumed to be identical. Any departure from this steady state is considered noise. The steady state is assumed to be stable in the sense that the pulse energy and timing fluctuations are damped and do not become very large (this implies no pulse dropouts). The pulse fluctuations are assumed to be stationary. In the steady state each pulse is assumed to obey the noiseless master equation with the active modulation frequency Ω_m equal to Ω_N. As before, we assume that the steady-state amplitude of each pulse is $\sqrt{n_p/\tau_N}B(t/\tau_N)$, where τ_N is the pulse width and n_p is the number of photons in each pulse. $B(t/\tau_N)$ is real and is normalized such that $\int_{-\infty}^{\infty}\mathrm{d}x\ B^2(x) = 1$.

If the timing fluctuations in different pulses are assumed to evolve completely independently, the finite-difference equation for the timing noise in each pulse can be obtained in the same way as for fundamentally mode-locked lasers. We should point out here that the energy fluctuations in different pulses cannot be assumed to evolve independently, since all the pulses interact with the same gain (and loss) medium, whose recovery time can be much longer than the pulse repetition time. For the timing fluctuations, assuming independent evolution, the N separate finite difference equations for the timing noise in N different pulses can be written in the compact form

$$\Delta t[n + N] - \Delta t[n] = -\gamma_N T_R\Delta t[n] + F_N[n], \tag{B13}$$

where γ_N is

$$\gamma_N = \frac{M}{2T_R}\Omega_N^{\,2}\tau_N^{\,2}\left\{\frac{1}{2\int_{-\infty}^{\infty}\mathrm{d}x\quad B'(x)B'(x)}\right\}. \tag{B14}$$

The noise source $F_N[n]$ represents the total timing noise added to each pulse in one round trip and has the correlation function

$$\langle F_N[n]F_N[m]\rangle = D_N\delta_{n,m}\,, \tag{B15}$$

where

$$D_N = \left(\frac{n_{\mathrm{sp}}}{n_p}\right)\langle L\rangle\tau_N^{\,2}\left\{\frac{1}{2\int_{-\infty}^{\infty}\mathrm{d}x\quad B'(x)B'(x)}\right\}. \tag{B16}$$

The expression for the mean square timing jitter in a harmonically mode-locked laser was given in Eq. (21),

$$\sigma_t^{\,2} = \langle\Delta t^2[n]\rangle = D_N/2\gamma_N T_R\,. \tag{B17}$$

Using the values for γ_N and D_N given above, we obtain

$$\sigma_t^{\,2} = \left(\frac{n_{\mathrm{sp}}}{n_p}\right)\langle L\rangle\frac{1}{M\Omega_N^{\,2}}. \tag{B18}$$

The pulse round-trip loss $\langle L\rangle$ is related to the photon lifetime τ_p in the laser cavity, $\langle L\rangle = T_R/\tau_p$. As in the fundamentally mode-locked case, the mean square timing jitter is found to be independent of the pulse shape and the pulse width, but the timing noise spectral density depends on both the pulse width and the pulse shape through γ_N [see relations (23) and (B14)]. When a fundamentally mode-locked laser is harmonically mode locked by means of increasing the modulation frequency from Ω_R to Ω_N, Eqs. (B12) and (B18) show that the mean square timing jitter decreases as long as the pulse energy, the modulation strength, and the cavity round-trip loss do not change. Also, the pulse shape and the pulse width may or may not change, depending on whether the dominant pulse shaping mechanism is dynamic gain or loss saturation or active modulation.

ACKNOWLEDGMENTS

The authors acknowledge helpful discussions with Paul W. Juodawlkis and Jeffrey J. Hargreaves. This work was supported by DARPA-PACT (Defense Advanced Research Projects Agency Photonics Analog-to-Digital Converter Technologies) program.

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