Estimating the Information Flow in Deep Neural Networks

Ziv Goldfeld

MIT

Information in the Modern Age & Complex Systems

Complex System: Multi-component system driven by local interactions

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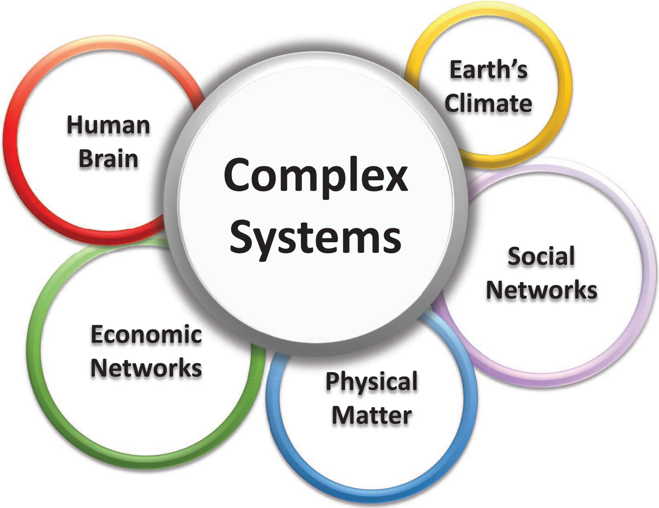

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Communication

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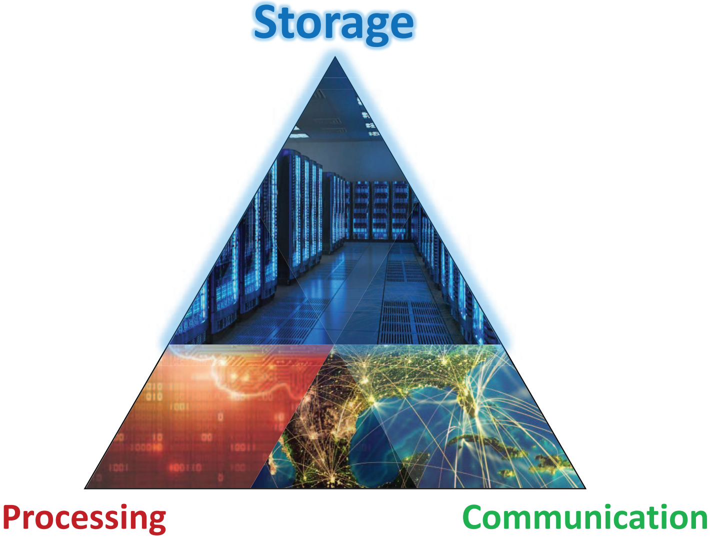

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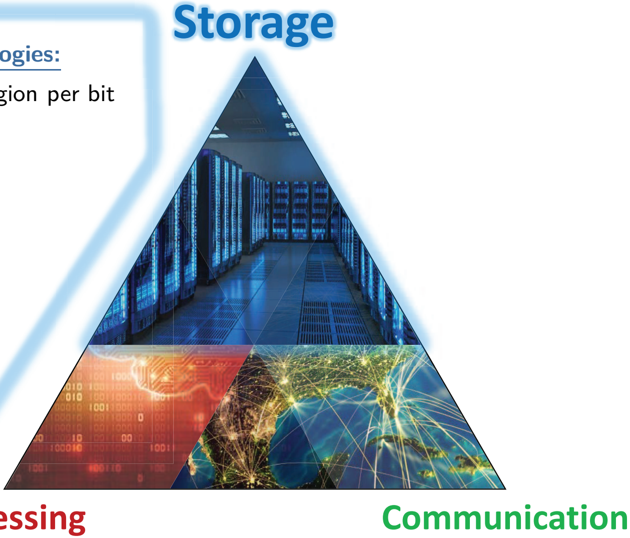
Emerging Technologies:

Shrink magnetic region per bit



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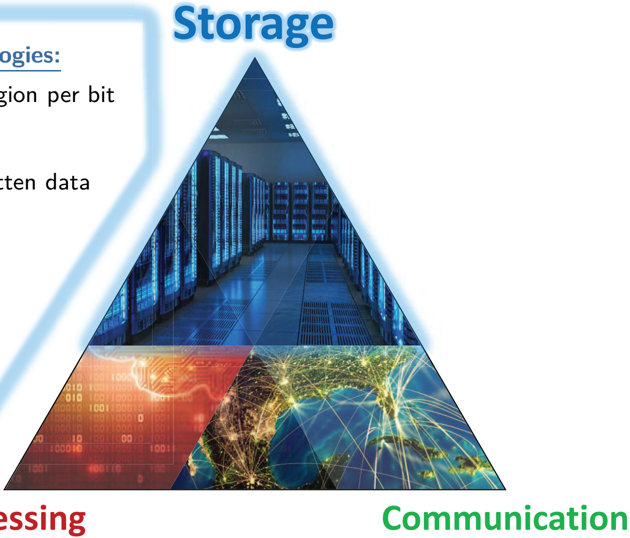

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Model & Study:

Interacting particle sys.

$\Downarrow$

Storage capacity &

HDD designs

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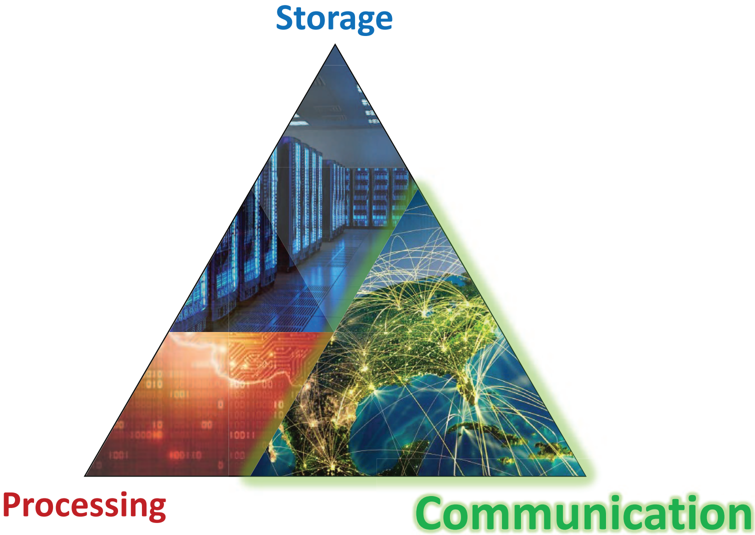

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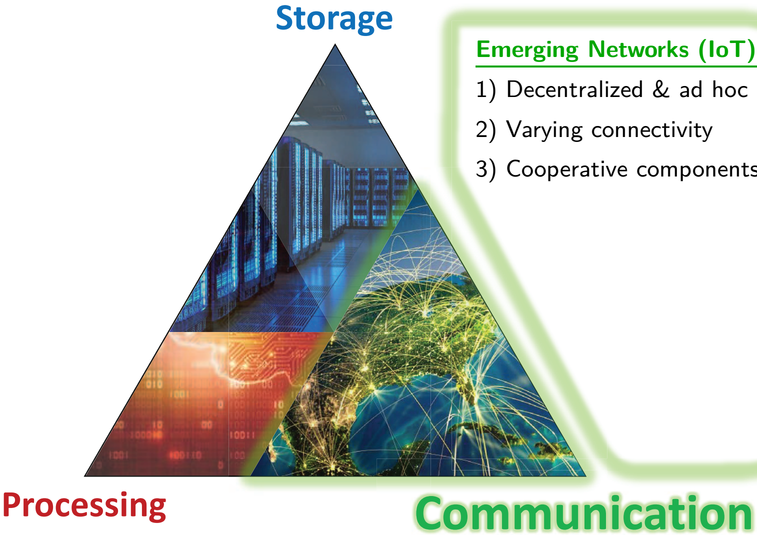

Emerging Networks (IoT):

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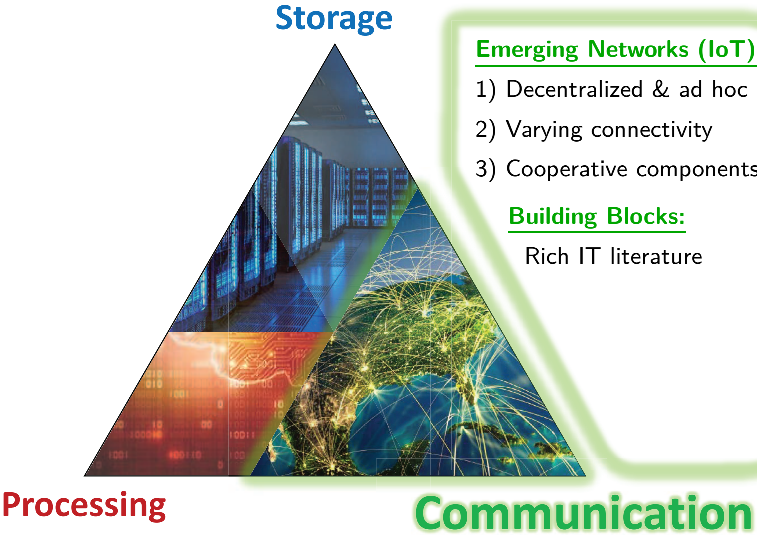

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Building Blocks:

Rich IT literature

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Network Modeling:

Random graphs & Actions based on primitives

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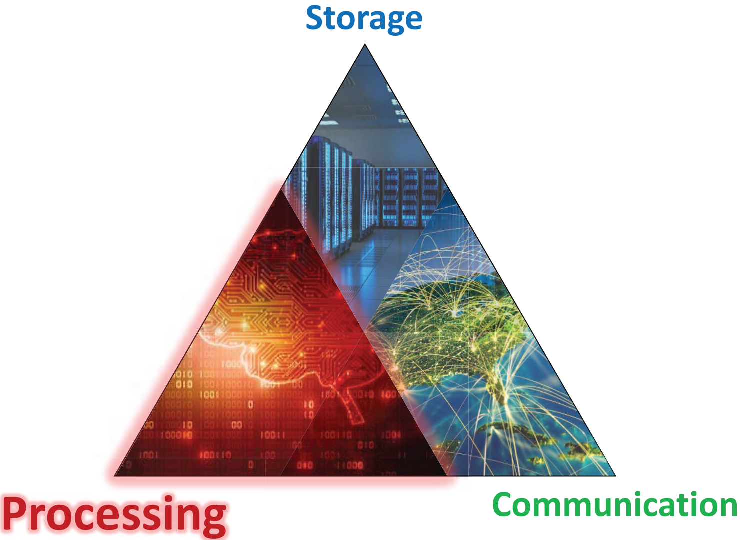

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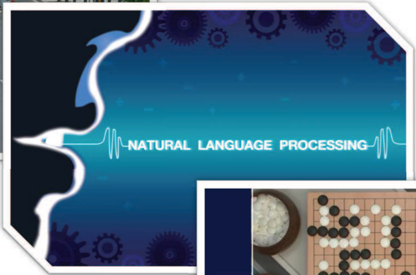

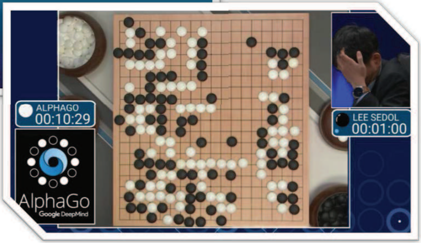

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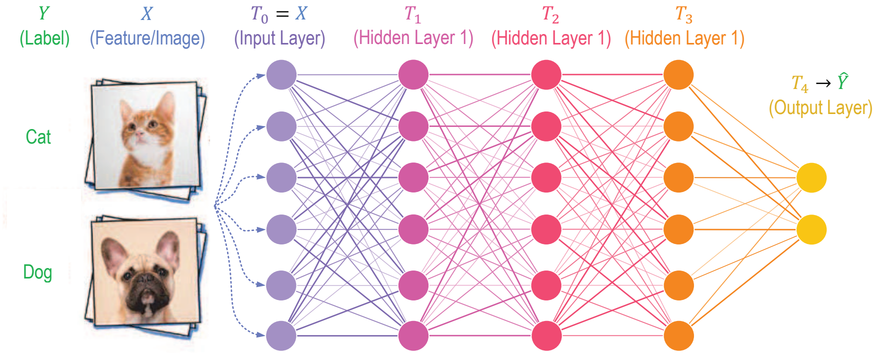
★ **Goal:** Explain 'compression' in Information Bottleneck framework

Setup and Preliminaries

Feedforward DNN for Classification:



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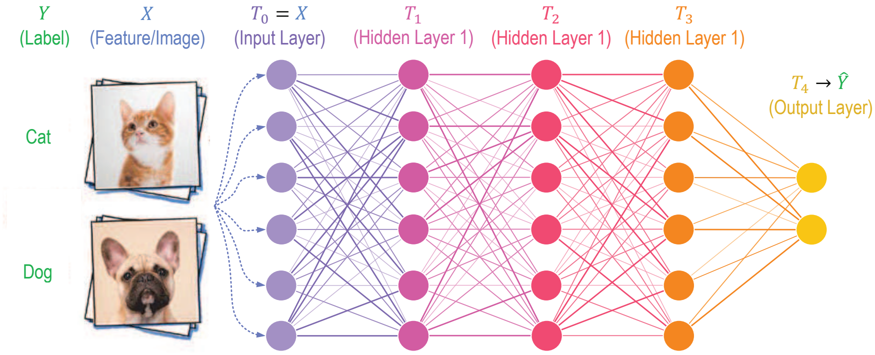

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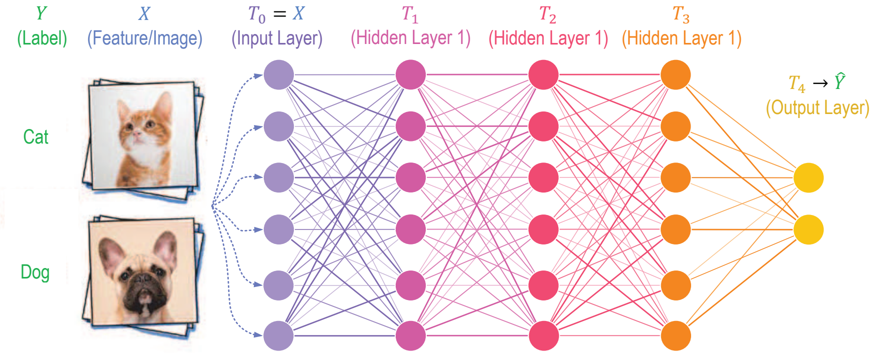

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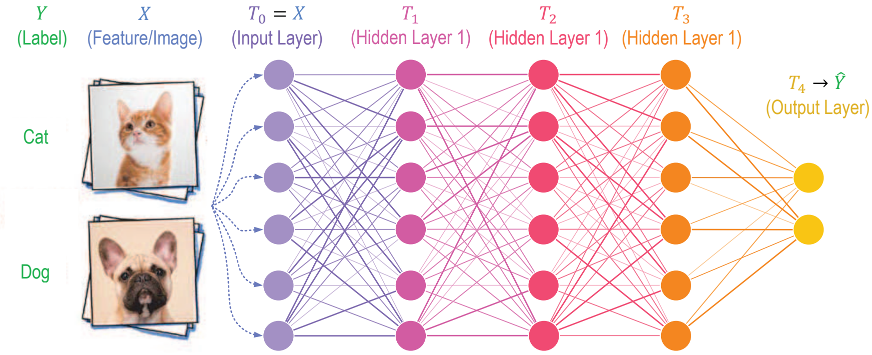

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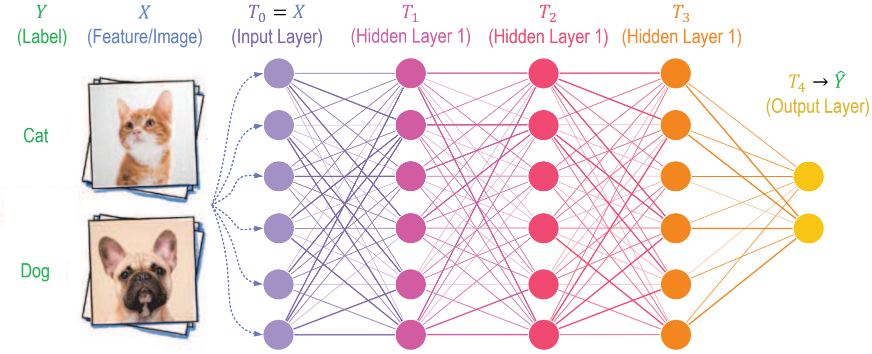

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- **IB Theory:** Track MI pairs $\big(I(X;T_\ell), I(Y;T_\ell)\big)$ (information plane)

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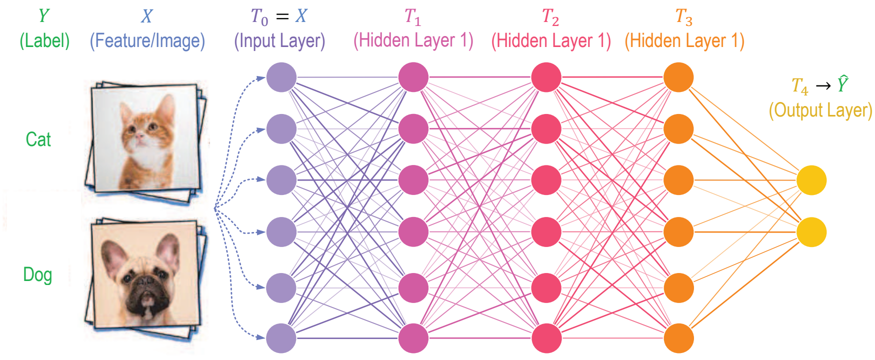

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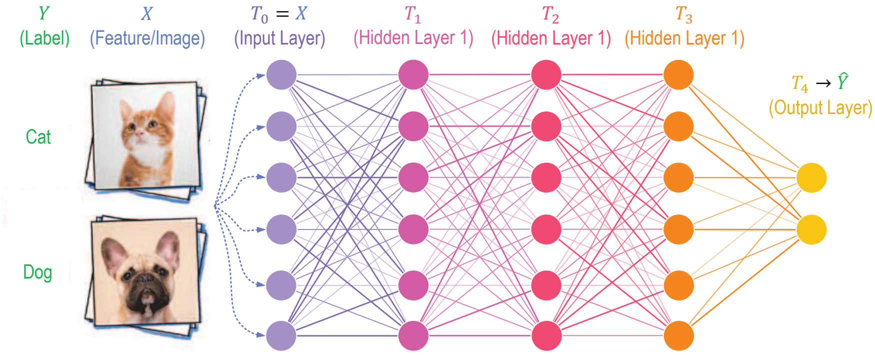

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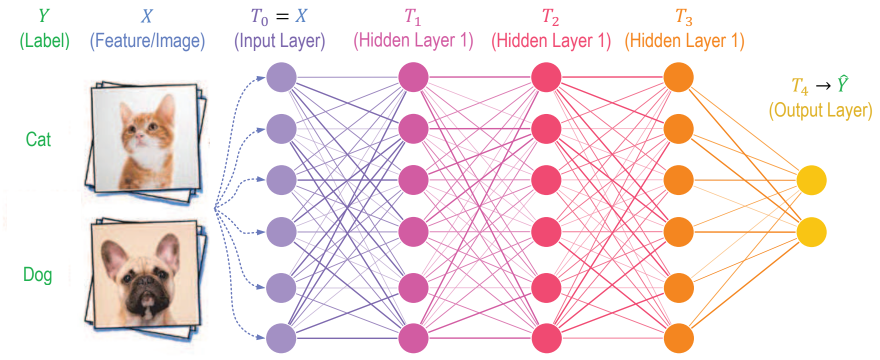

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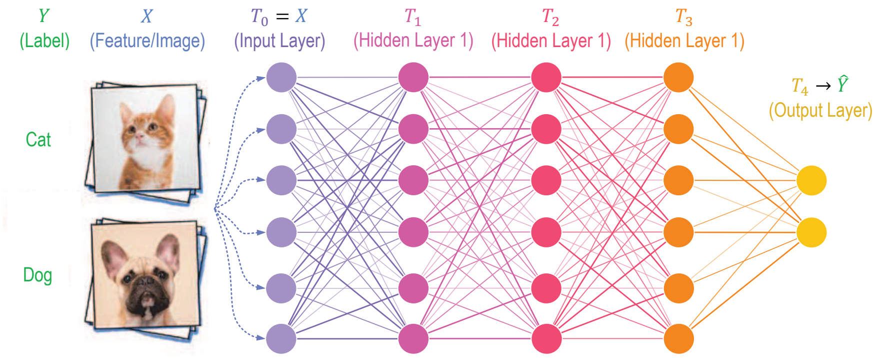
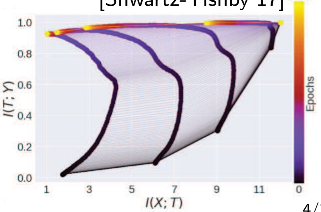


[Shwartz-Tishby'17]



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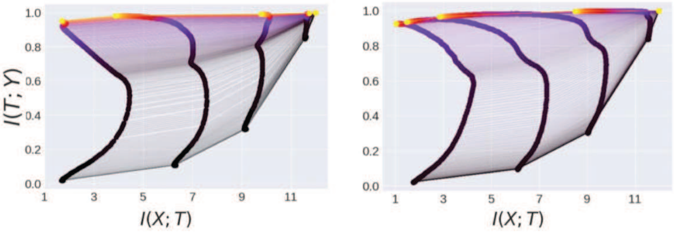
Past Works:

[Shwartz-Tishby'17, Saxe *et al.*'18]



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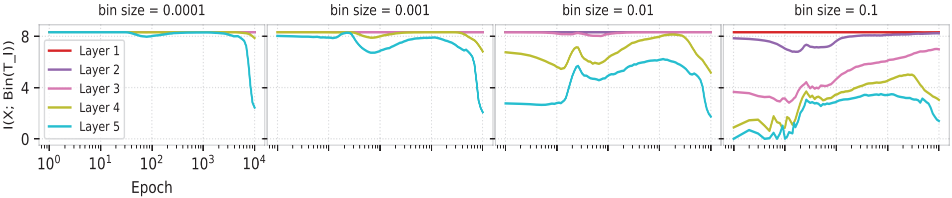

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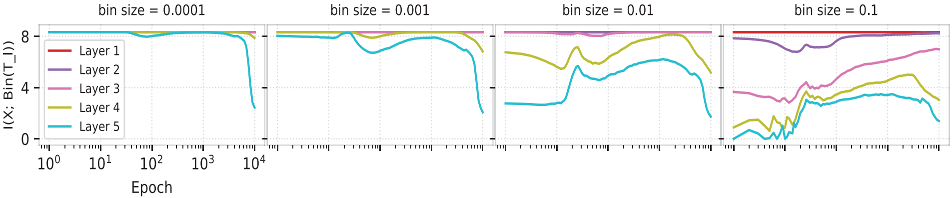

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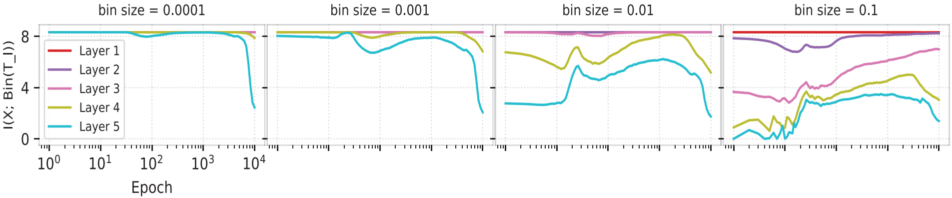

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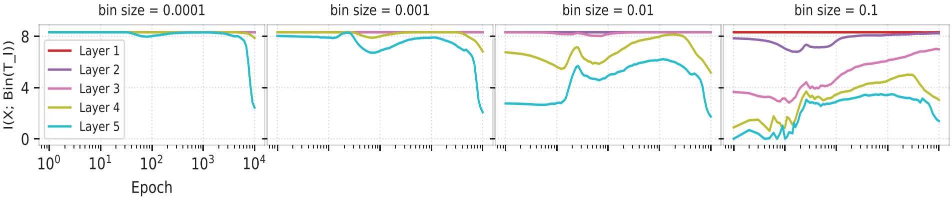

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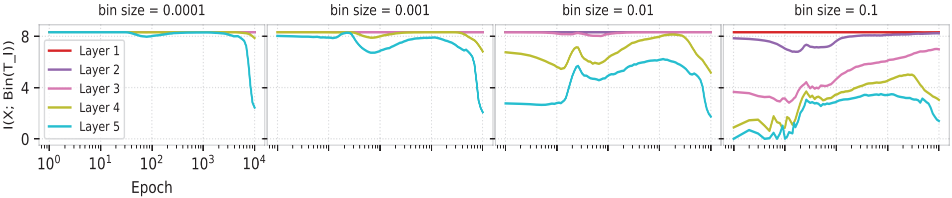

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⊛ **Real Problem:** $I(X;T_\ell)$ is meaningless in det. DNNs

Auxiliary Framework - Noisy Deep Neural Networks

Modification: Inject (small) Gaussian noise to neurons' output

[G.-Berg-Greenewald-Melnyk-Nguyen-Kingsbury-Polyanskiy'18]

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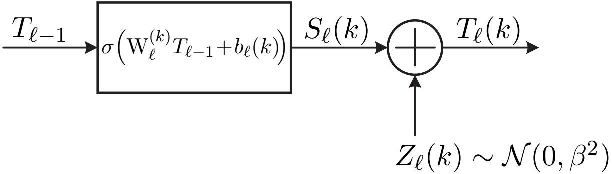

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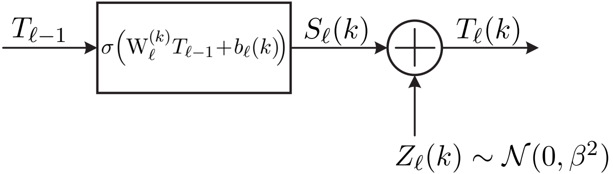

$\Longrightarrow$ $X \mapsto T_\ell$ is a **parametrized channel** that depends on DNN param.!

Auxiliary Framework - Noisy Deep Neural Networks

Modification: Inject (small) Gaussian noise to neurons' output

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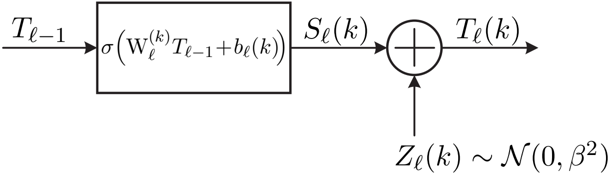

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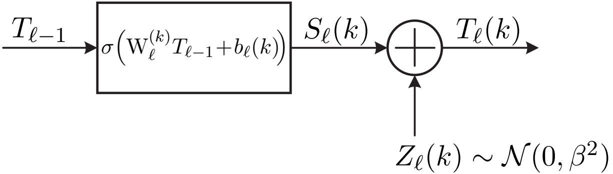

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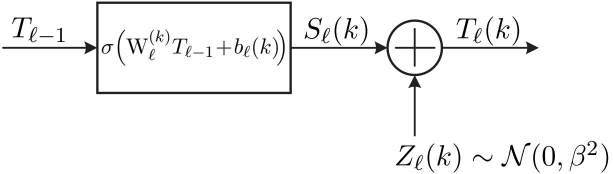

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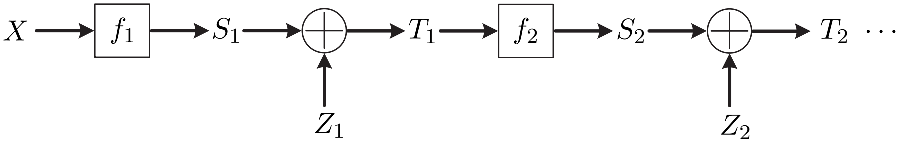
- **Operational Perspective:**

  Performance & learned representations similar to det. DNNs ($\beta \approx 10^{-1}$)

Mutual Information in Noisy DNNs

Mutual Information Estimation in Noisy DNNs

Noisy DNN:



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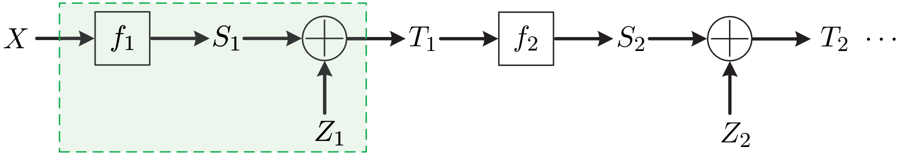

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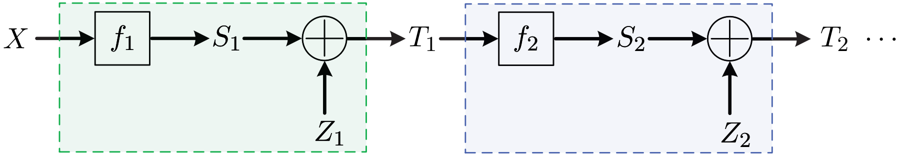

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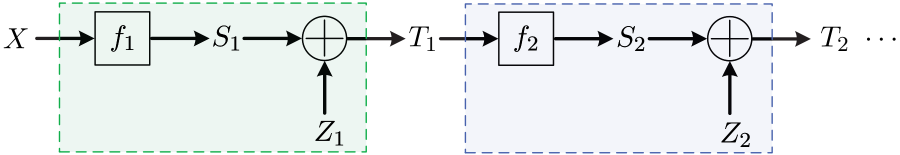

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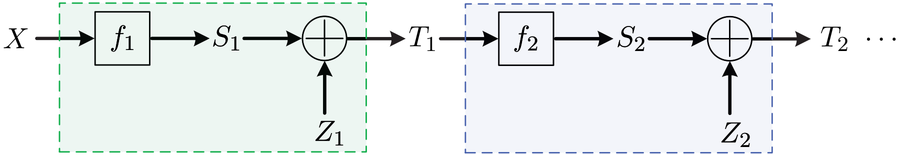

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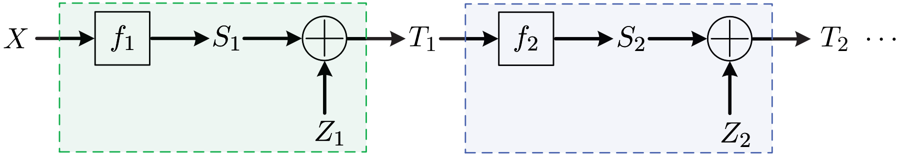

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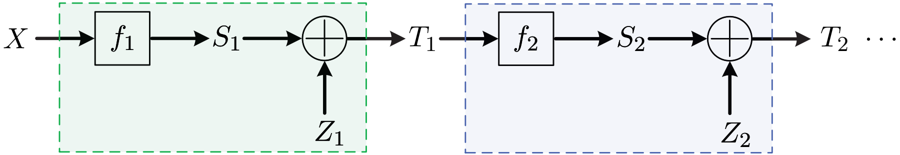

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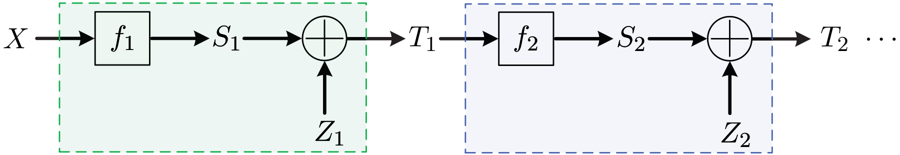

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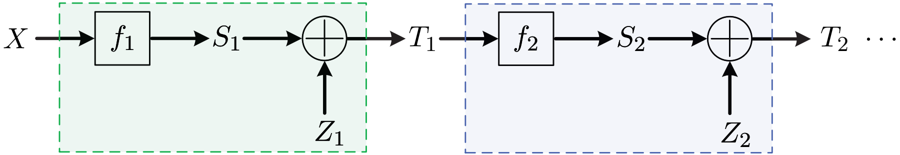

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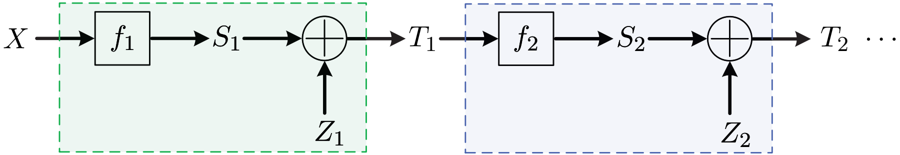

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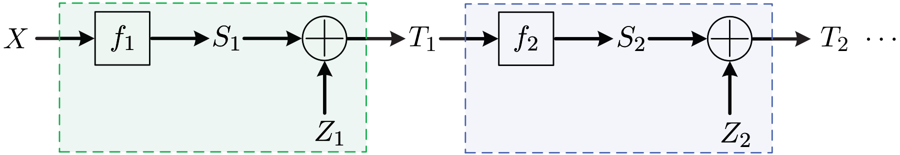

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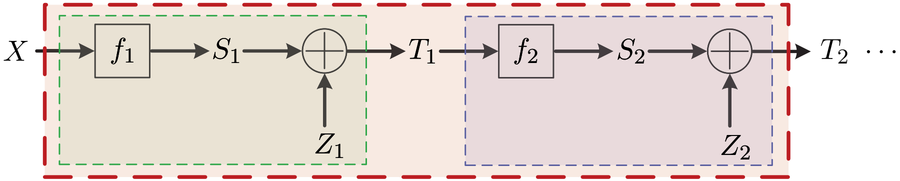

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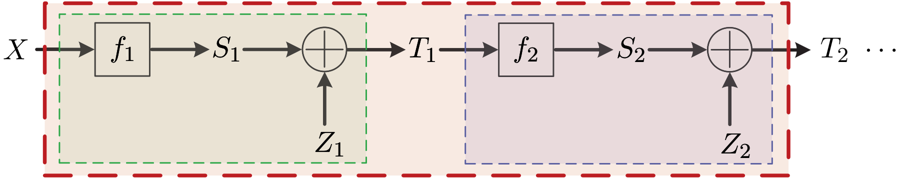

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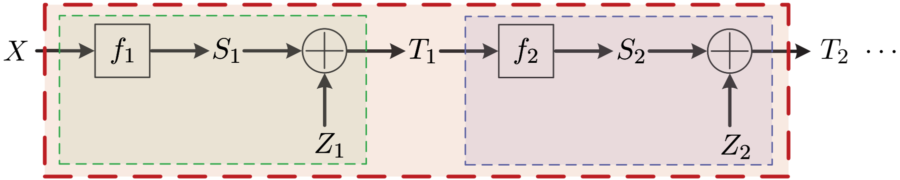

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⊛ But both are **easily sampled** via the DNN forward pass

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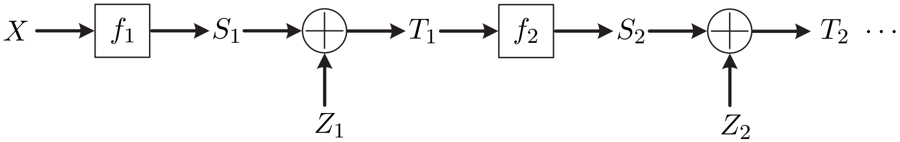

Differential Entropy Estimation under Gaussian Convolutions

Estimate $h(P * \varphi)$ based on n i.i.d. samples from $P \in \mathcal{F}_d$ (nonparametric class) and knowledge of φ (PDF of $\mathcal{N}(0, \beta^2 \mathrm{I}_d)$).

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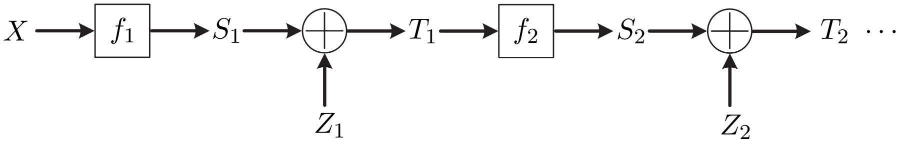

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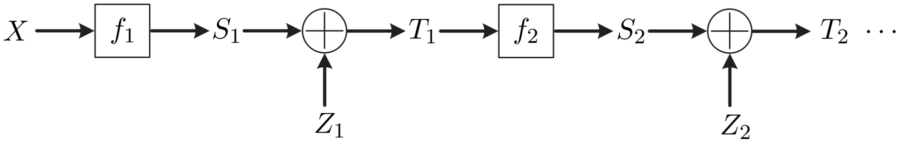

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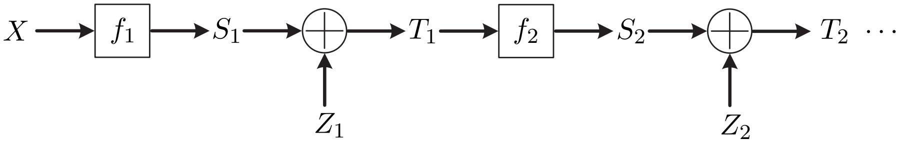

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- **Guarantees:** Estimation risk is $O(1/\sqrt{n})$ (all constants explicit)*

$\star$ Exponentially large in d though constants, which is provably necessary.

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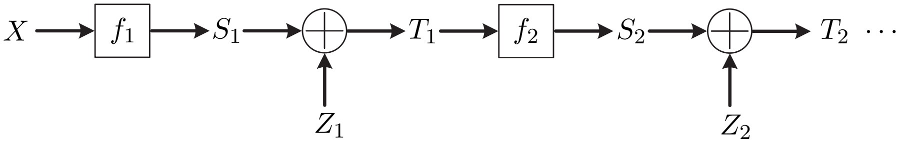

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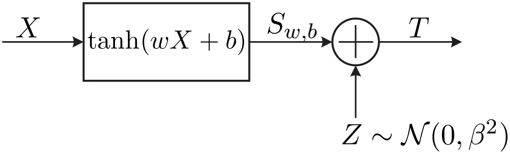
- **Guarantees:** Estimation risk is $O(1/\sqrt{n})$ (all constants explicit)*
- **Faster Rate:** kNN/KDE est. via 'noisy' samples attain $O\left(n^{-\frac{a}{b+d}}\right)$

Back to Noisy DNNs

$I(X;T_\ell)$ Dynamics - Illustrative Minimal Example

Single Neuron Classification:



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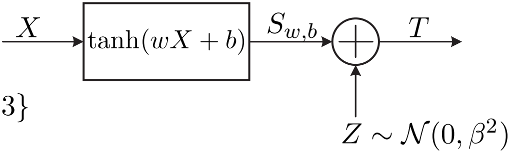

- **Input:** $X \sim \mathsf{Unif}\{\pm 1, \pm 3\}$

  $\mathcal{X}_{y=-1} \triangleq \{-3,-1,1\}$, $\mathcal{X}_{y=1} \triangleq \{3\}$

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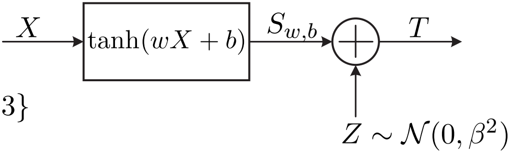

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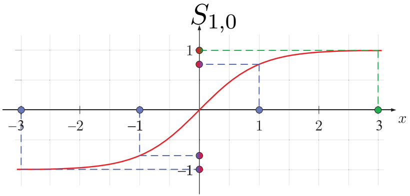

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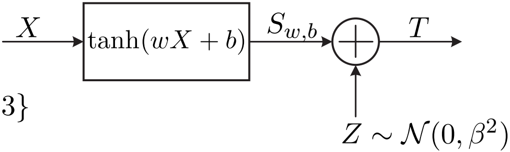

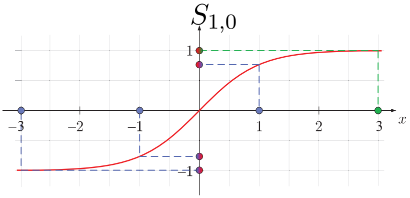

⊛ Center & sharpen transition ($\iff$ increase w and keep $b=-2w$)

$I(X;T_\ell)$ Dynamics - Illustrative Minimal Example

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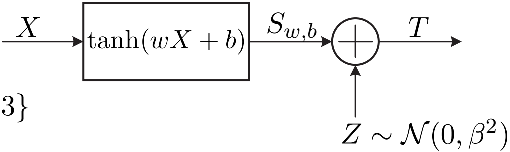

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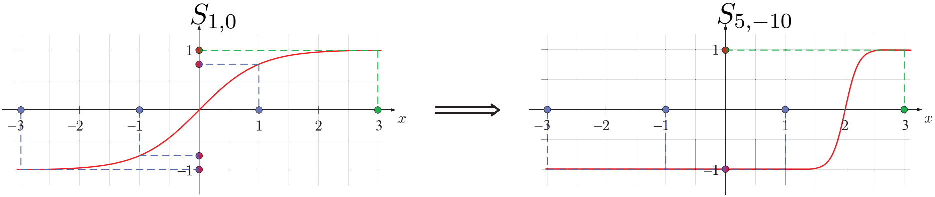

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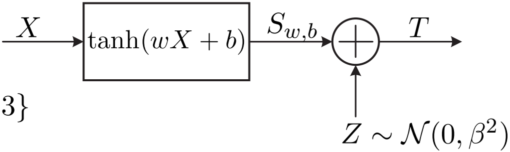

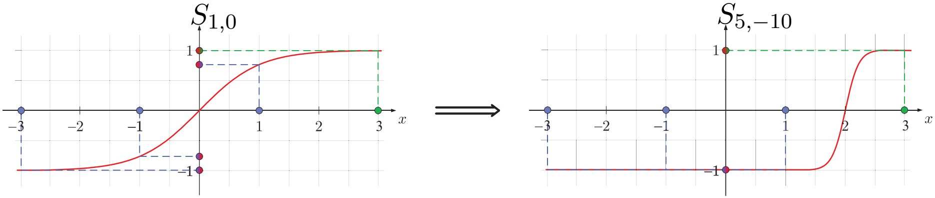

✓ Correct classification performance

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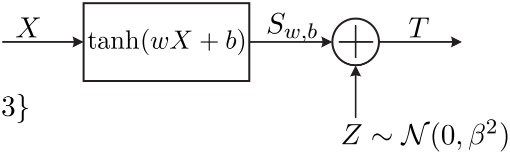

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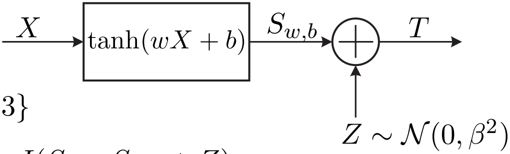

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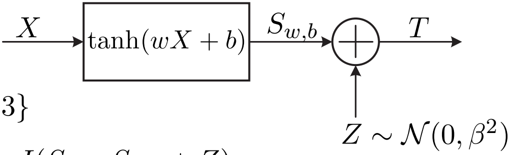
$I(X;T_\ell)$ Dynamics - Illustrative Minimal Example



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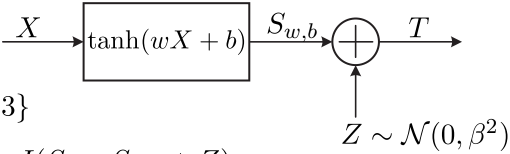

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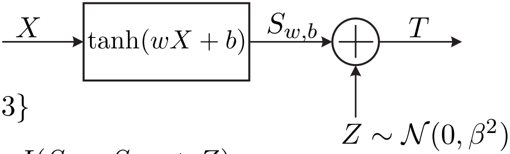

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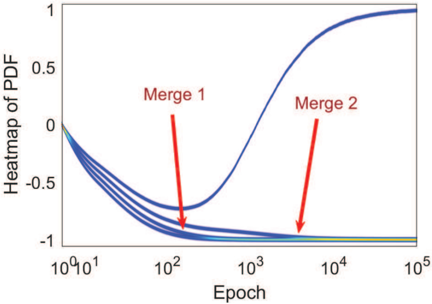

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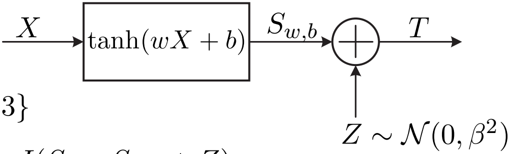

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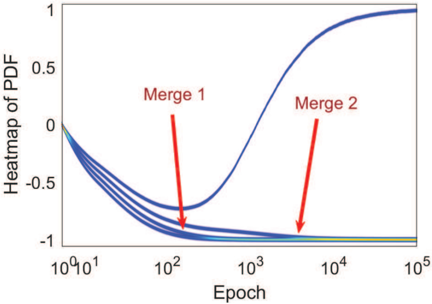

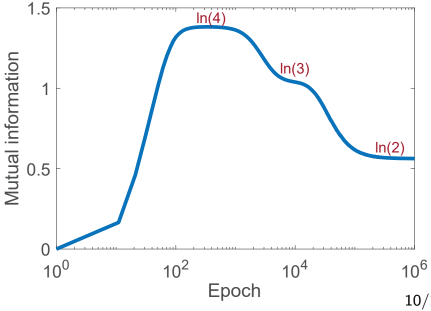

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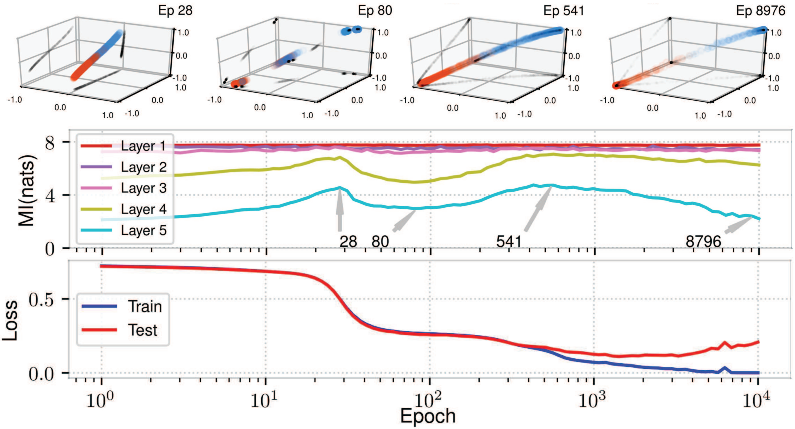

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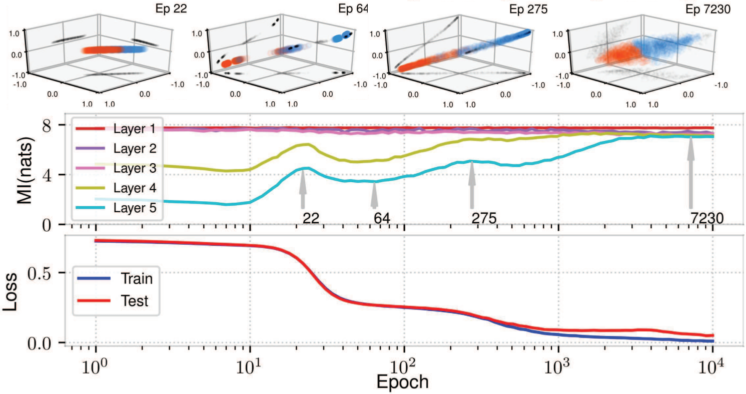

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$\Longrightarrow$ Compression of $I(X;T_\ell)$ driven by clustering of representations

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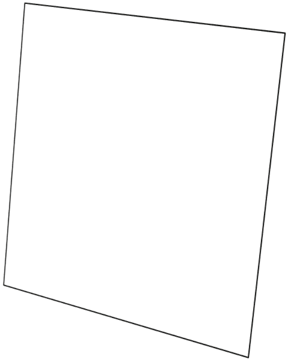

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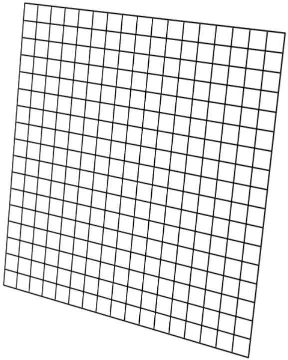

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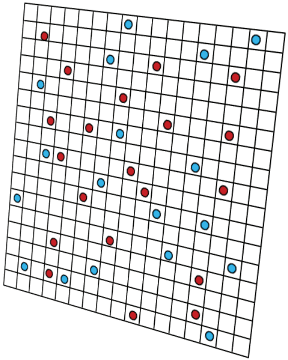

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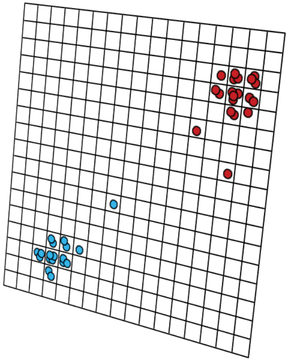

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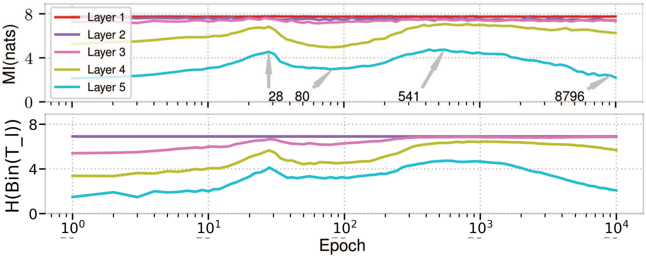

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$\implies$ Past works were not showing MI but clustering (via binned-MI)!

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$\Longrightarrow$ **Clustering** is the common phenomenon of interest!

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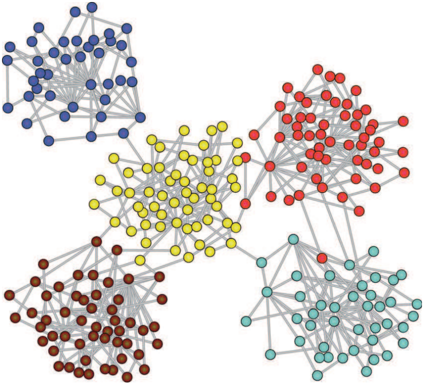

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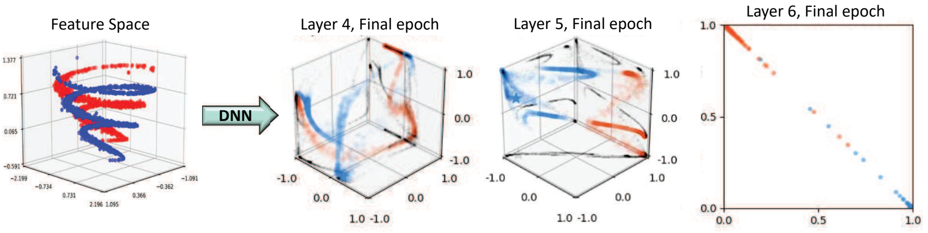

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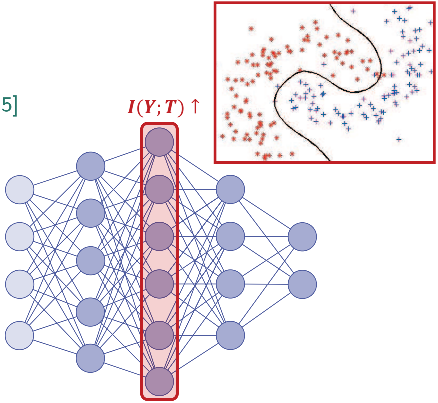

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[1] Z. Goldfeld, E. van den Berg, K. Greenewald, I. Melnyk, N. Nguyen, B. Kingsbury and Y. Polyanskiy, "Estimating information flow in neural networks," Arxiv preprint `https://arxiv.org/abs/1810.05728`, October 2018.

[2] Z. Goldfeld, K. Greenewald and Y. Polyanskiy, "Estimating differential entropy under Gaussian convolutions," Submitted to the *IEEE Transactions on Information Theory*, October 2018.
Arxiv: `https://arxiv.org/abs/1810.11589`

[3] Z. Goldfeld, G. Bresler and Y. Polyanskiy, "Information storage in the stochastic Ising model," Submitted to the *IEEE Transactions on Information Theory*, May 2018.
Arxiv: `https://arxiv.org/abs/1805.03027`

Information Storage in Interacting Particle Systems

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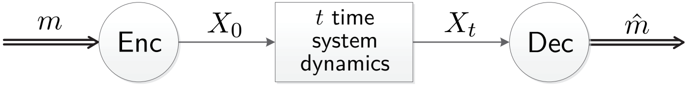

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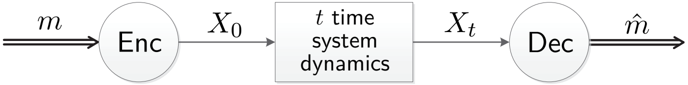

[G.-Bresler-Polyanskiy'18] Performance benchmarks & hard-drive designs

Cooperative Communication Networks

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Q: Reliable (& secure) information passing protocols? Fundamental limits?

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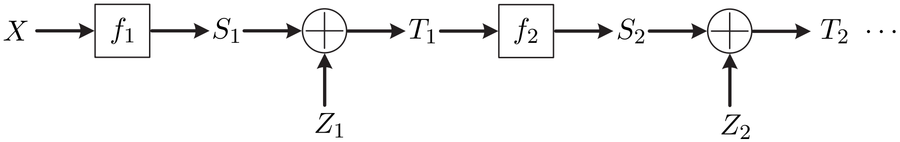

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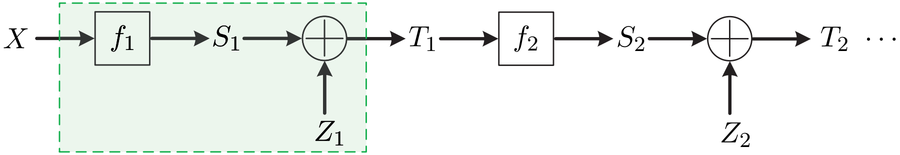

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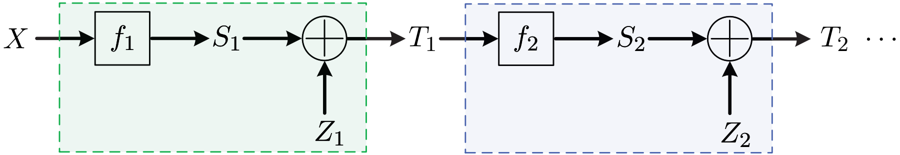

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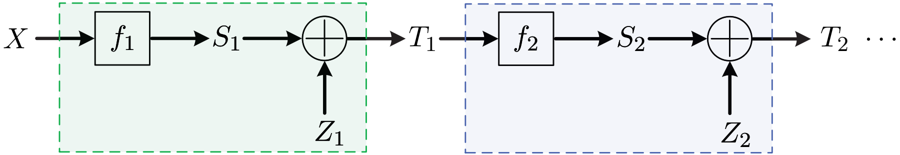

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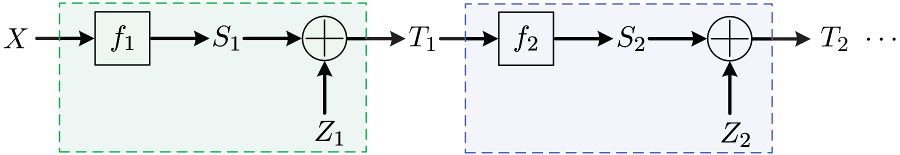

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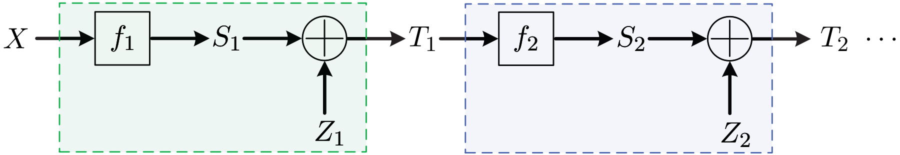

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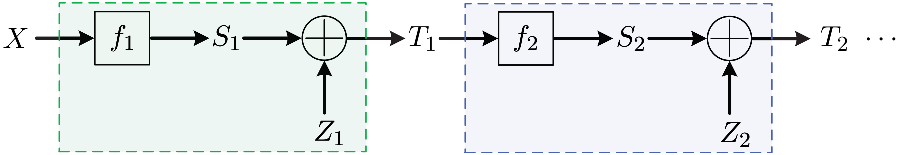

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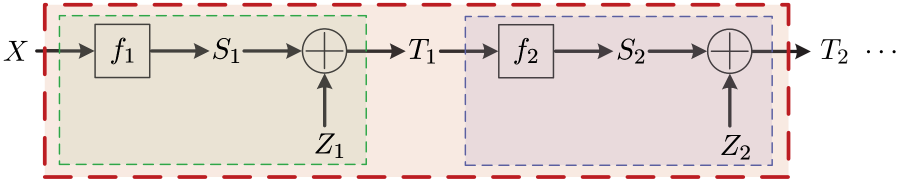

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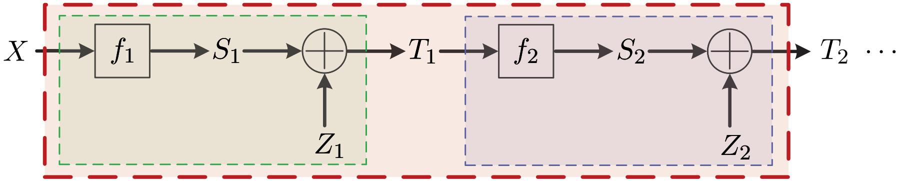

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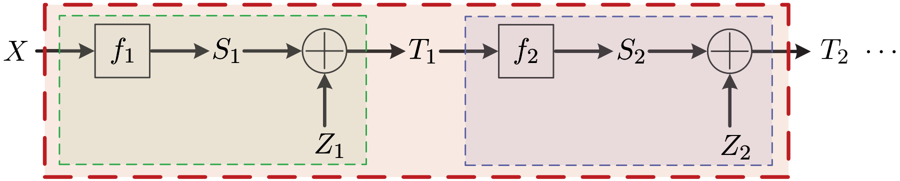

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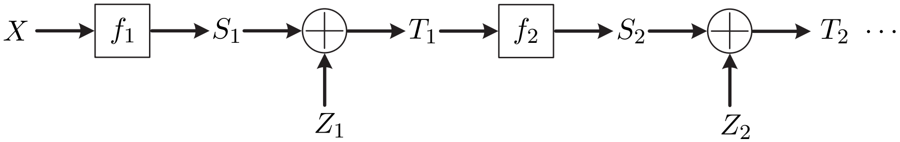

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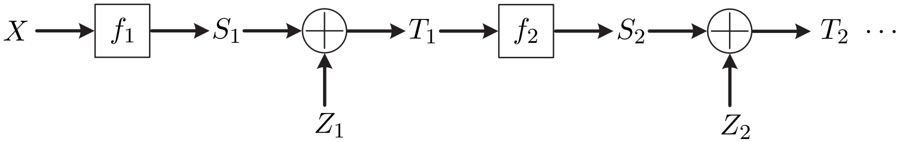

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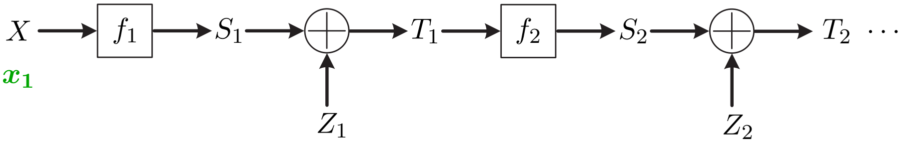

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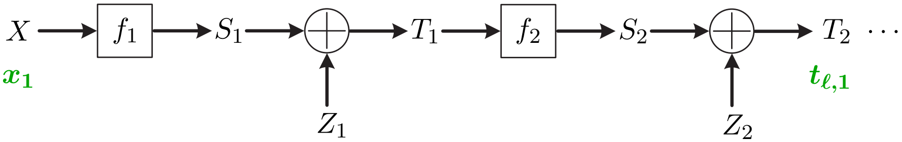

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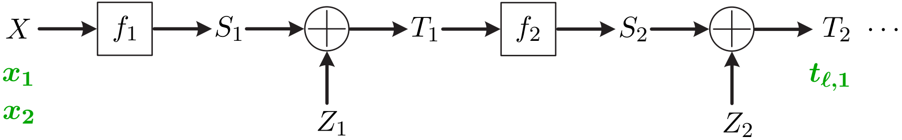

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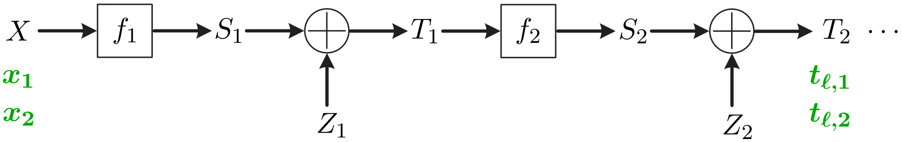

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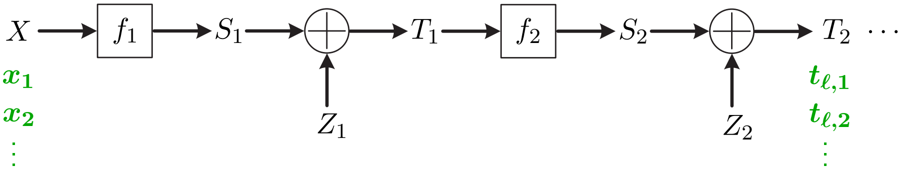

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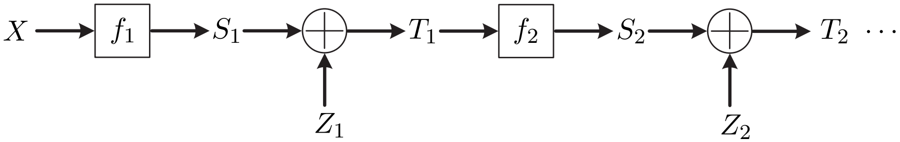

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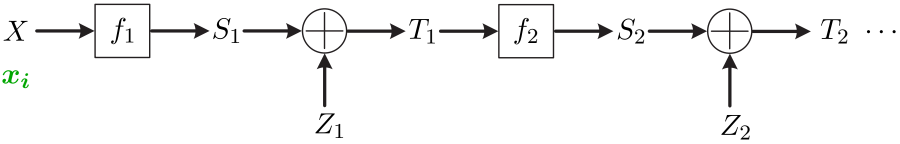

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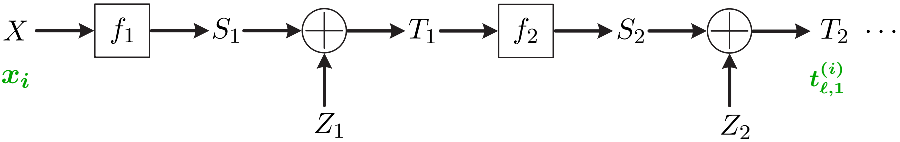

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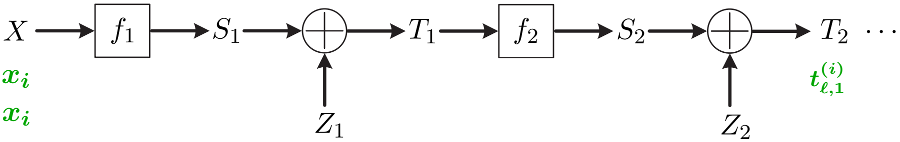

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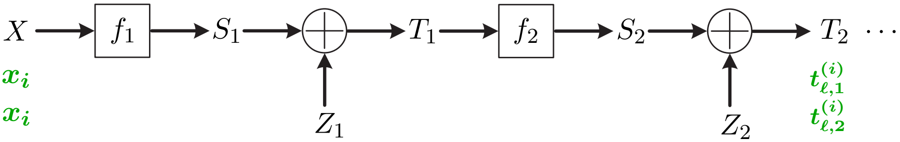

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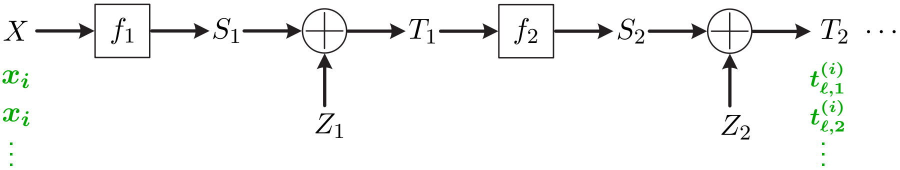

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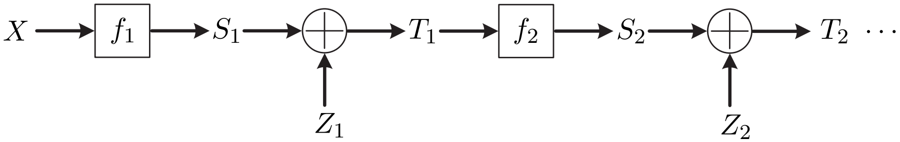

$\Longrightarrow$ Estimate $I(X;T_\ell)$ from samples via **general-purpose $h(P)$ est.:**

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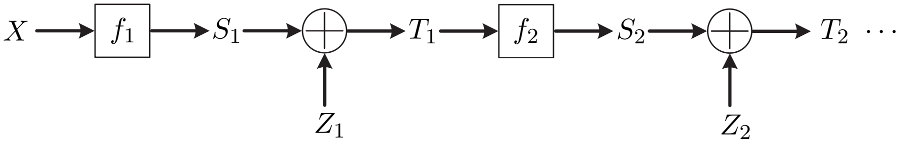

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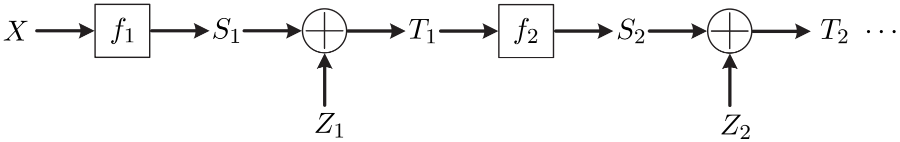

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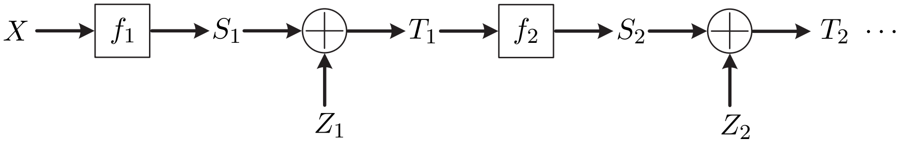

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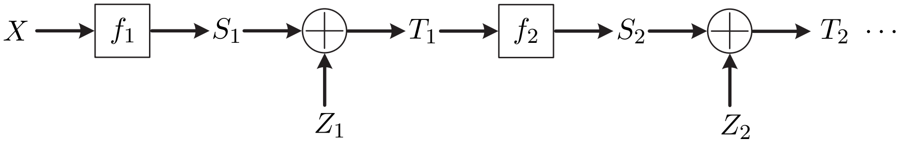

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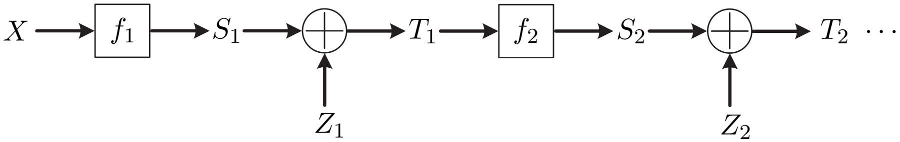

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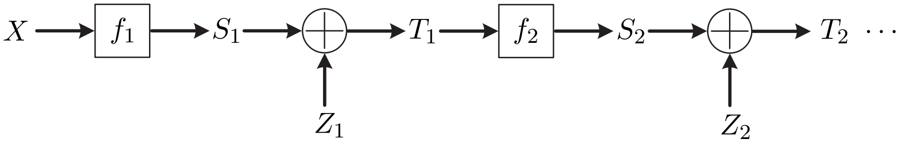

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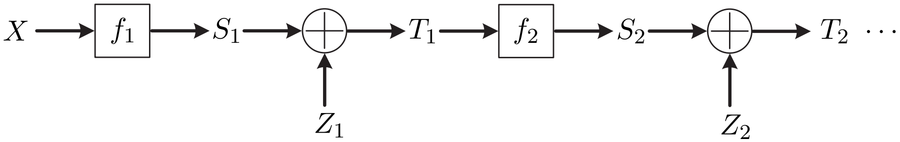

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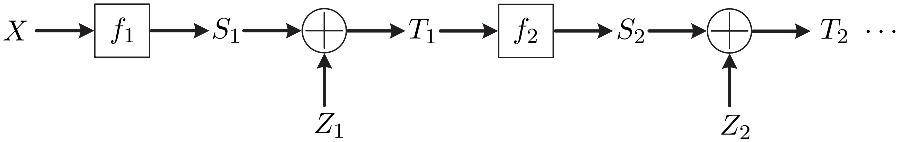

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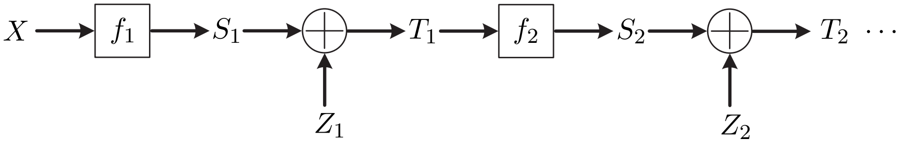

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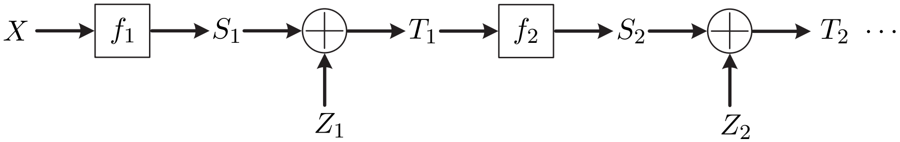

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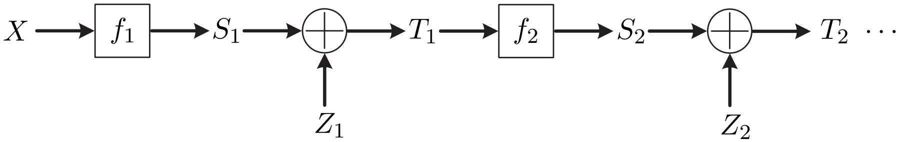

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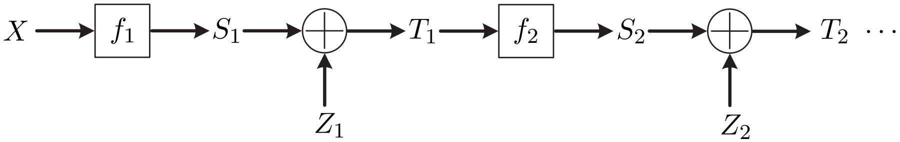

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Estimate $h(P * \varphi)$ based on n i.i.d. samples from $P \in \mathcal{F}_d$ (nonparametric class) and knowledge of φ (PDF of $\mathcal{N}(0, \beta^2 \mathrm{I}_d)$).

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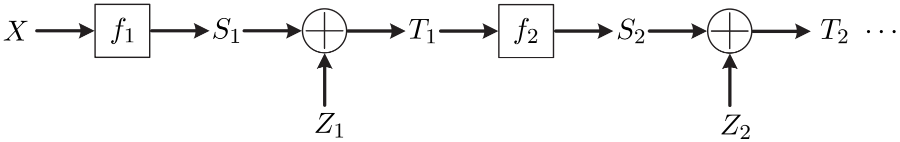

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Nonparametric Class: Depends on DNN architecture (nonlinearities)

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Abs. Error Minimax Risk: S^n are n i.i.d. samples from P, define

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Theorem (ZG-Greenewald-Polyanskiy '18)

For $\mathcal{F}_d \triangleq \left\{P \middle| \mathsf{supp}(P) \subseteq [-1,1]^d\right\}$ *and any* $\beta > 0$ *and* $d \geq 1$, *we have*

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$$+\left(c_{\beta,d}^2 + \frac{2c_{\beta,d}d(1+\beta^2)}{\beta^2} + \frac{8d(d+2\beta^4+d\beta^4)}{\beta^4}\right)\frac{2}{n}$$

where $c_{\beta,d} \triangleq \frac{d}{2}\log(2\pi\beta^2) + \frac{d}{\beta^2}$.

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- **Extension:** P with sub-Gaussian marginals (ReLU + Weight regular.)

The Sample Propagation Estimator - Proof Ideas

Strategy: Split analysis to $\mathcal{R} \triangleq [-1,1]^d + \mathcal{B}(0, \sqrt{c \log n})$ and $\mathcal{R}^c$

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The Sample Propagation Estimator - Proof Ideas

Strategy: Split analysis to $\boldsymbol{\mathcal{R} \triangleq [-1,1]^d + \mathcal{B}(0, \sqrt{c\log n})}$ and $\boldsymbol{\mathcal{R}^c}$

- **Restricted Entropy:** $h_{\mathcal{R}}(p) \triangleq \mathbb{E}\left[-\log p(X)\mathbb{1}_{\{X\in\mathcal{R}\}}\right]$

$$\sup\mathbb{E}\left|h(P*\varphi)-h(\hat{P}_n*\varphi)\right| \leq \sup\mathbb{E}\left|h_{\mathcal{R}}(P*\varphi)-h_{\mathcal{R}}(\hat{P}_n*\varphi)\right|+2\sup\left|h_{\mathcal{R}^c}(P*\varphi)\right|$$

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- **Inside $\boldsymbol{R}$:** ▶ $-t\log t$ modulus of cont. for $x \mapsto x\log x$ & Jensen's ineq.

$$\implies \text{Focus on analyzing} \quad \mathbb{E}\left|(P*\varphi)(x) - (\hat{P}_n*\varphi)(x)\right|$$

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  - Bias & variance analysis

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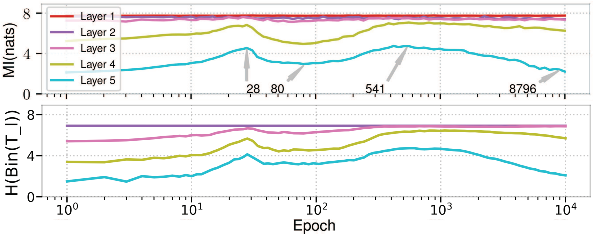
- **Outside R:** $O\left(\frac{1}{n}\right)$ decay via Chi-squared distribution tail bounds

Binning vs True Mutual Information

Comparing to Previously Shown MI Plots:



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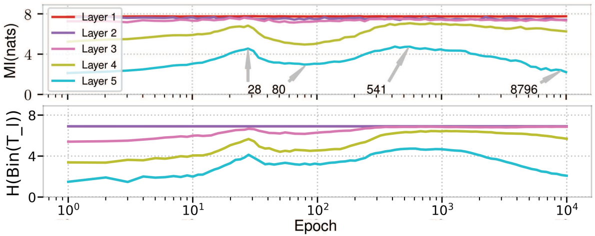

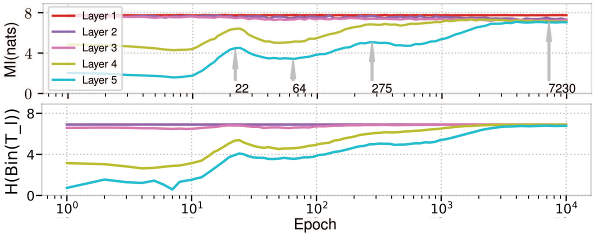

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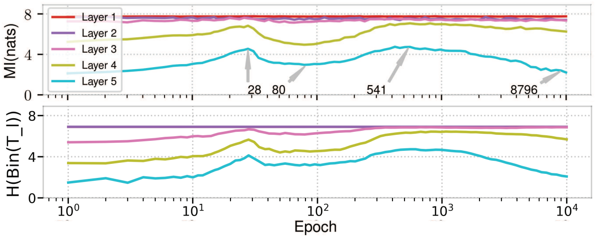

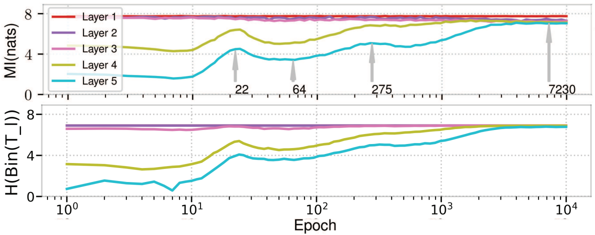

$\Longrightarrow$ Past works were not showing MI but clustering (via binned-MI)!