Information Storage in the Stochastic Ising Model at Zero Temperature

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MIT

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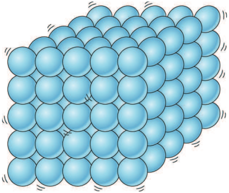
Storing Information Inside Matter



Storing Information Inside Matter

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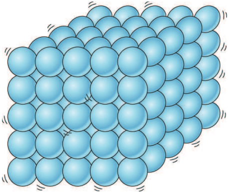

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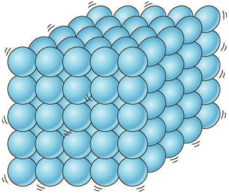

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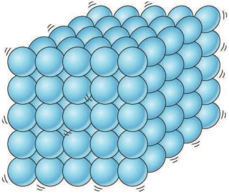

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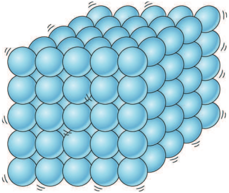

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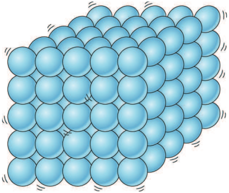

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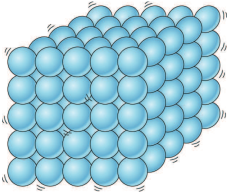

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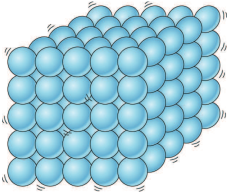

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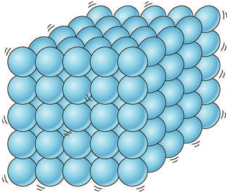

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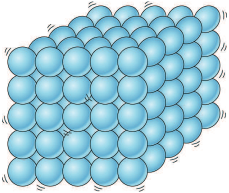
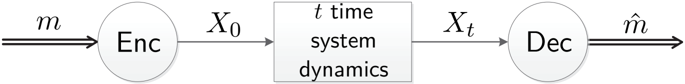


Goals:

- Distill notion of storage from particular technology
- Capture interparticle interaction and system's dynamics
- How much data can be stored and for how long?

Operational Framework



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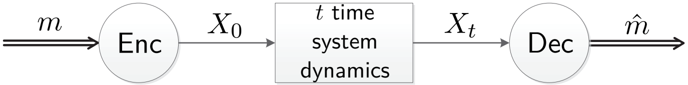

Stochastic Ising Model:

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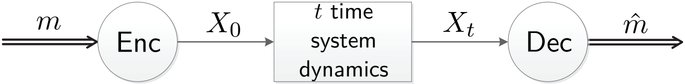

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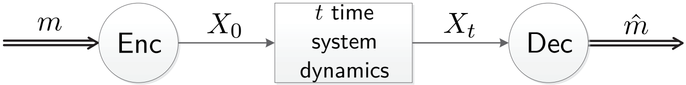

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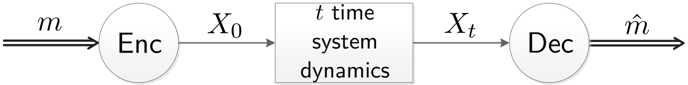

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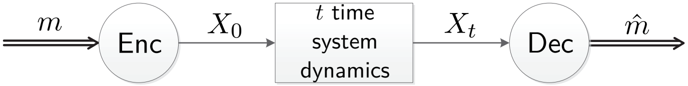

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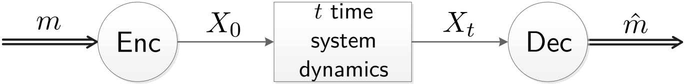

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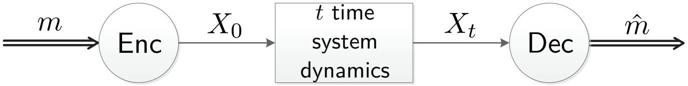

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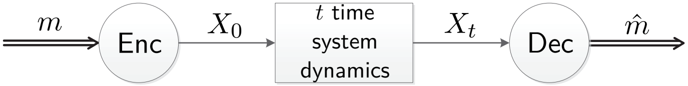

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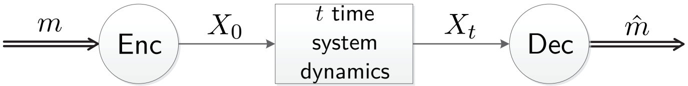

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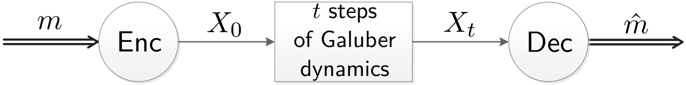
Warm (β small) $\implies$ Weak interactions

Cold (β large) $\implies$ Strong interactions

Measuring Information Storage



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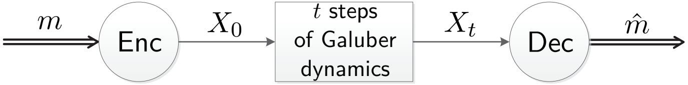

Information Capacity:

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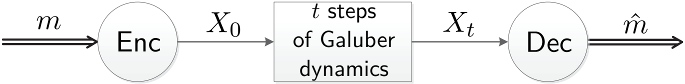

Information Capacity: $I_n(t) \triangleq \max_{P_{X_0}} I(X_0; X_t)$

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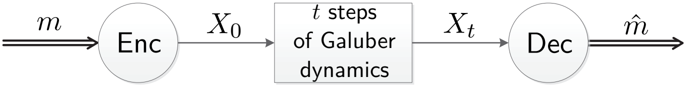

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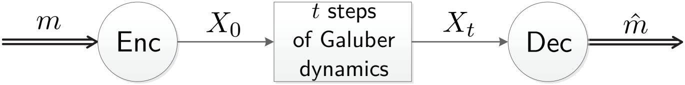

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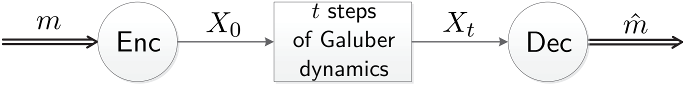

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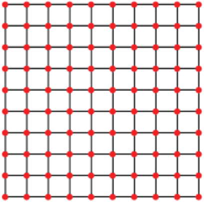

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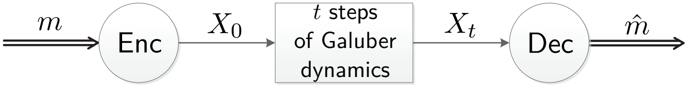

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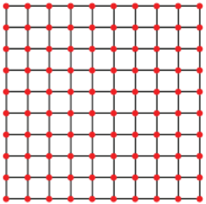

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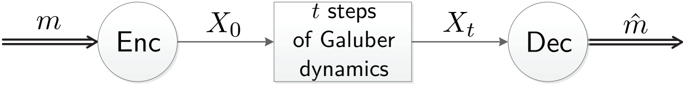

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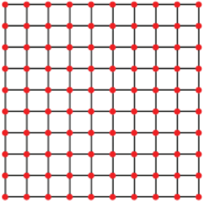

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⊛ **Cold:** Can interactions (memory) help?

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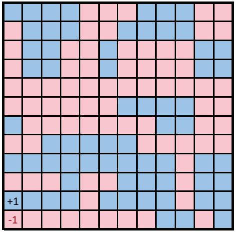

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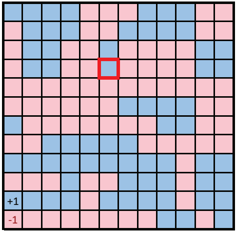

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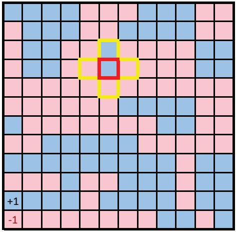

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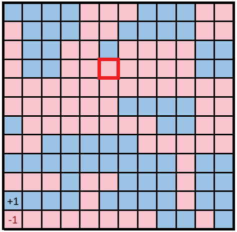

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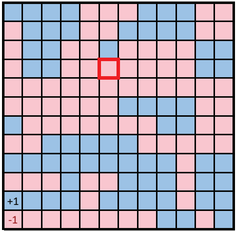

Time	Information Capacity	Comments
$t = 0$	$I_n(t) = n$	Upper bound $\forall t$
$t = O(n)$	$I_n(t) = \Theta(n)$	Constant 'physical' time
$t = a(n) \cdot n$ $a(n) \in \omega(1)$	$I_n(t) = \Omega\left(\frac{n}{a(n)}\right)$	$I_n(n \log n) = \Omega\left(\frac{n}{\log n}\right)$ $I_n(n^{1+\alpha}) = \Omega(n^{1-\alpha})$
$t \to \infty$ ind. of n	$I_n(\infty) = \Theta(\sqrt{n})$	Lower bound $\forall t$

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Theorem (Goldfeld-Bresler-Polyanskiy'19)

Fix $\epsilon \in \left(0, \frac{1}{2}\right), \gamma > 0$. *For* β *sufficiently large there exist* $c > 0$ *s.t.*

$$I(X_0; X_t) \le \log 2 + \epsilon_n(\beta),$$

for all $t \ge n \cdot e^{c\beta n^{\frac{1}{4}+\epsilon}}$, *where* $X_0 \sim \pi$ *and* $\lim_{n\to\infty} \epsilon_n(\beta) = 0$.

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$\Longrightarrow$ Storage beyond exponential time ≤ 1 bit ($X_0 \sim$Gibbs)

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Proposition (Martinelli'94)

Let ϵ, γ be as before. For β sufficiently large there exist $c > 0$ s.t.

$$\sum_{\substack{\sigma \in \Omega_n: \\ m(\sigma) > 0}} \pi(\sigma) \mathbb{P}\Big(X_t^\sigma \neq X_t^\boxplus\Big) \leq e^{-\gamma\sqrt{n}}, \quad \forall t \geq n \cdot e^{c\beta n^{\frac{1}{4}+\epsilon}}$$

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Proof Outline

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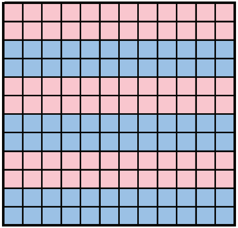

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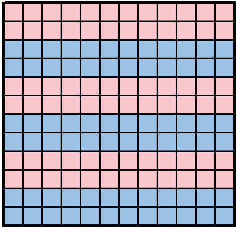

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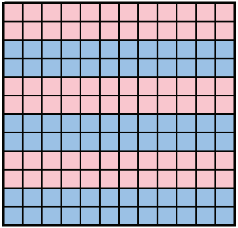

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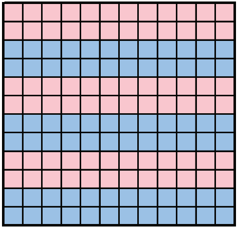

Reduction to Single Stripe Analysis

- **Denote:** $t_f \triangleq e^{c\beta}$; $X_t^{(j)} \triangleq X_t\Big|_{\text{Stripe } j}$; $X_t^{[j]} \triangleq \big(X_t^{(k)}\big)_{k=1}^{j}$

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$\Longrightarrow$ Suffices to analyze $\mathbb{P}$(More than half stripe flipped)

Single Stripe Case: Main Result

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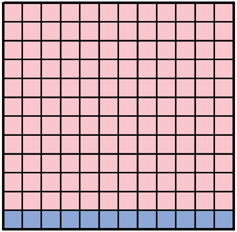

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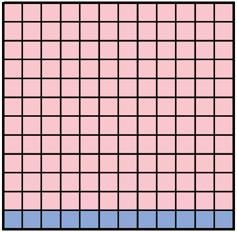

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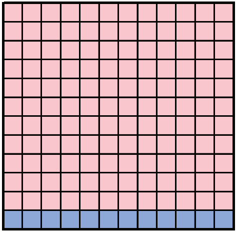

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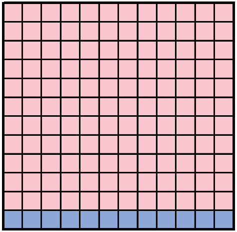

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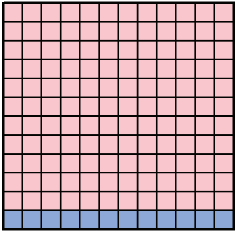

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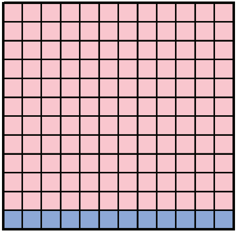

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Fix any $c, C \in (0,1)$. For β and n sufficiently large, we have

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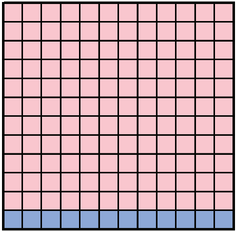

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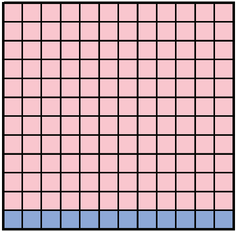

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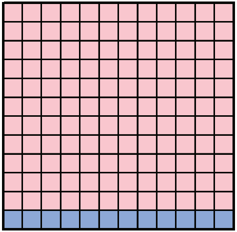

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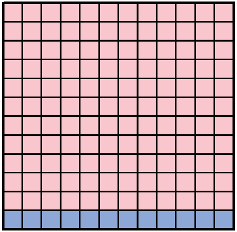

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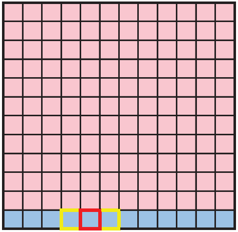

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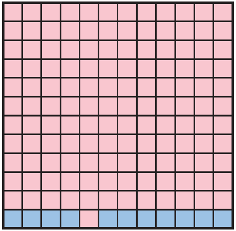

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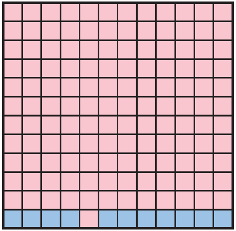

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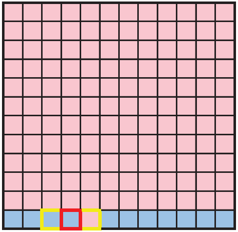

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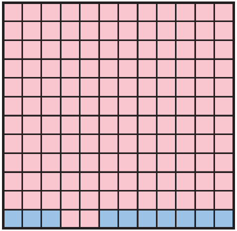

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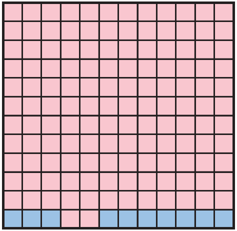

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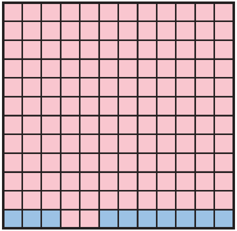

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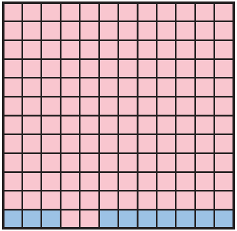

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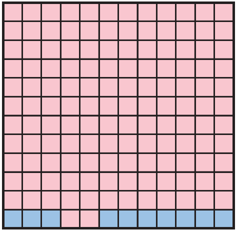

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$\Longrightarrow$ Dominate $\{X_t\}_t$ by a phase-separated dynamics

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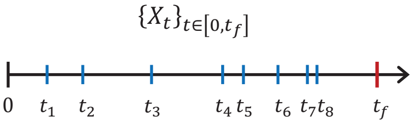

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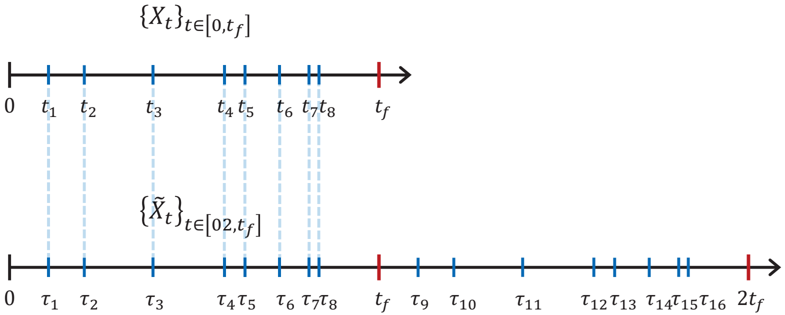
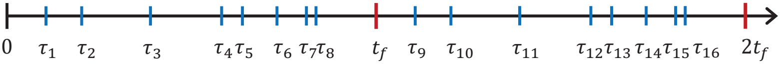
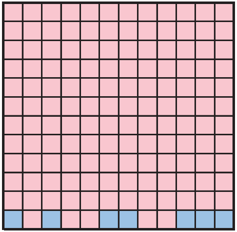


Single Stripe Case: Phase-Separated Dynamics (2)

$$\{\tilde{X}_t\}_{t\in[0,2t_f]}$$





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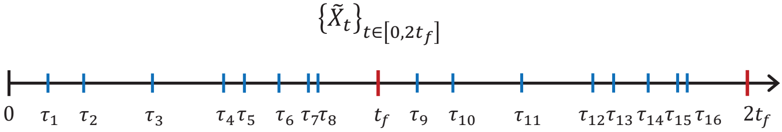
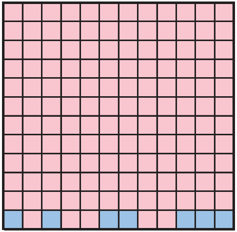


Blocking Rule:



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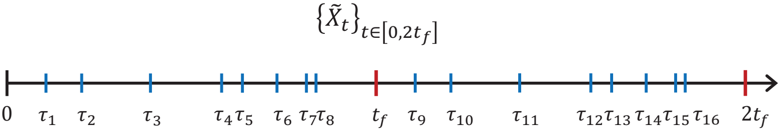

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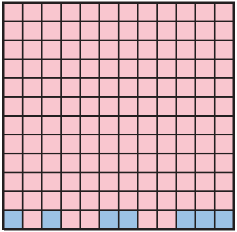

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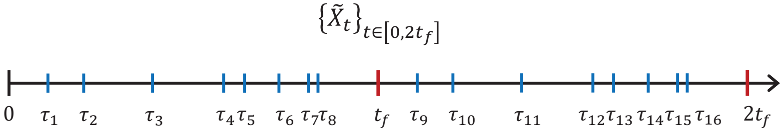

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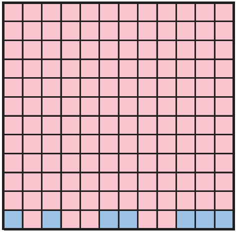

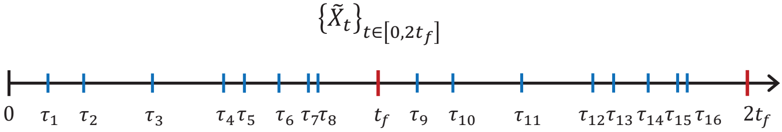

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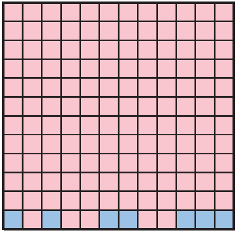

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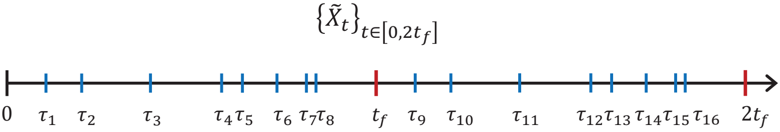

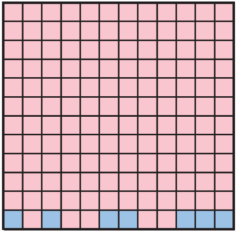

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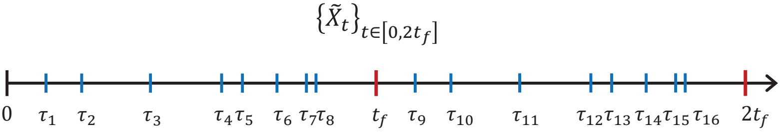

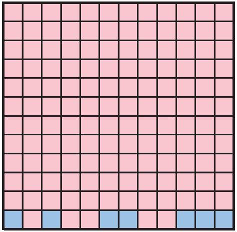

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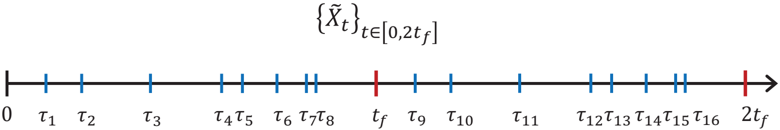

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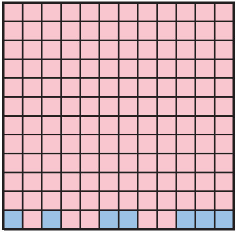

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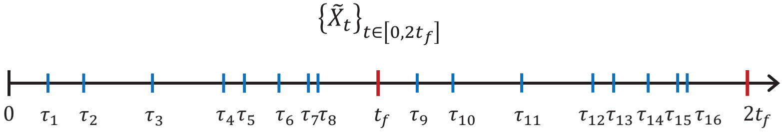

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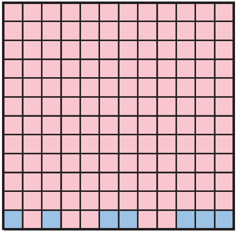

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$\Longrightarrow$ New dynamics is a speedup: $\mathbb{E}N^{(+)}(t_f) \geq \mathbb{E}\tilde{N}^{(+)}(2t_f)$

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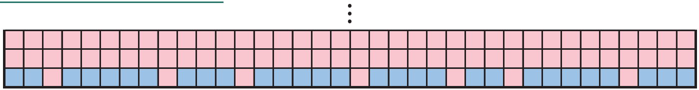

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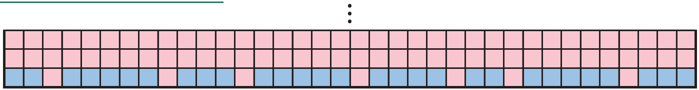

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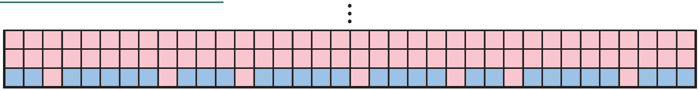

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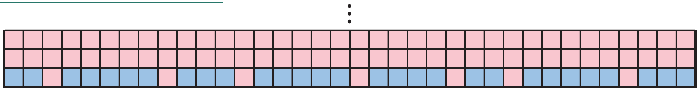

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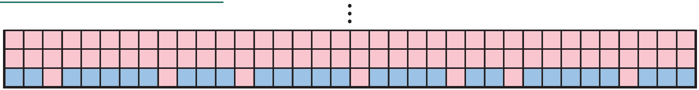

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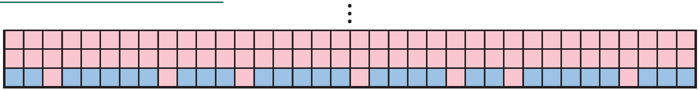

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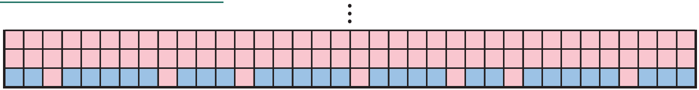

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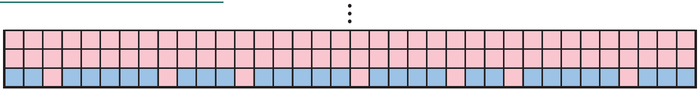

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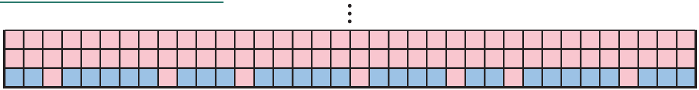

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