Information Storage Capacity of Interacting Particle Systems

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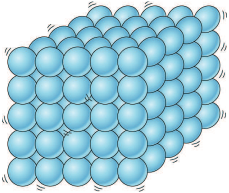
Cornell University

Collaborators: Guy Bresler and Yury Polyanskiy

Beyond IID in Information Theory 8

Nov. 13th, 2020

Storing Information Inside Matter



Storing Information Inside Matter

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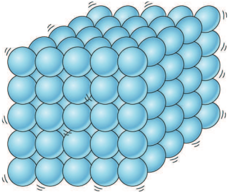

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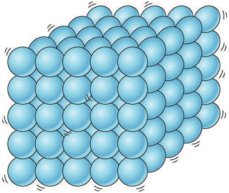

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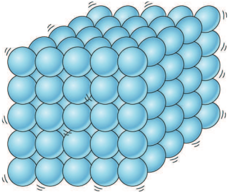

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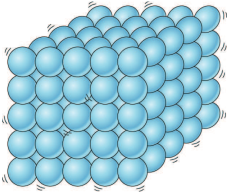

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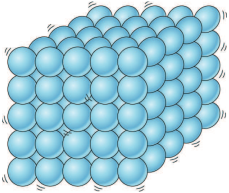

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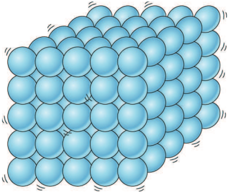

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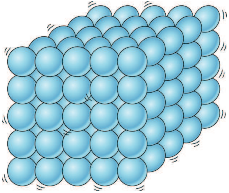

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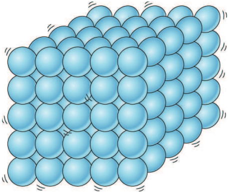

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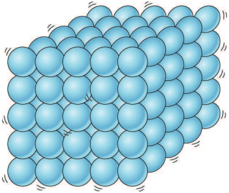

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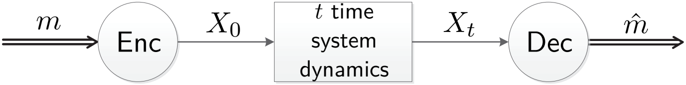
- Capturing interparticle interaction and system's dynamics
- How much data can be stored and for how long?

Operational Framework



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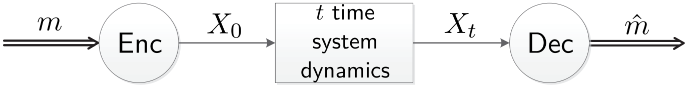
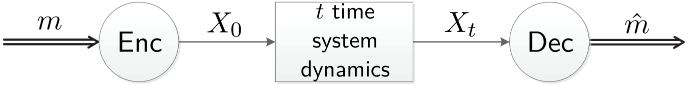


Stochastic Ising Model:

Operational Framework



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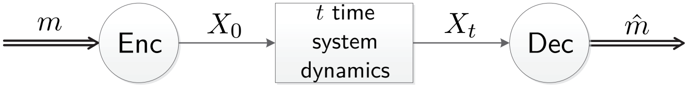

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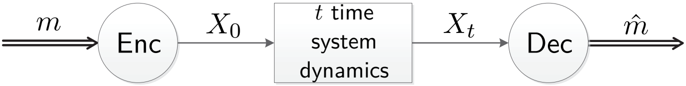

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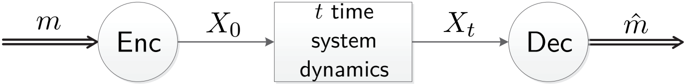

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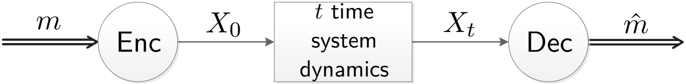

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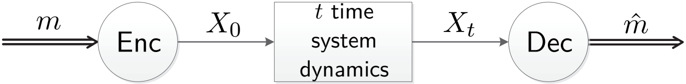

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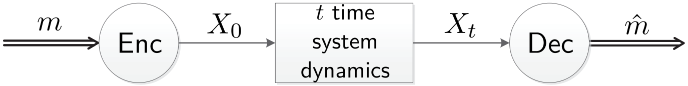

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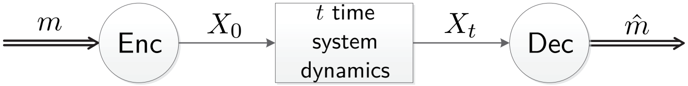

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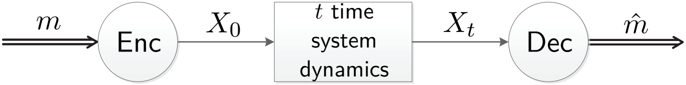

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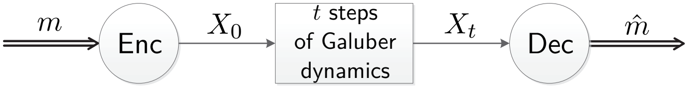
Warm (β small) $\implies$ Weak interactions

Cold (β large) $\implies$ Strong interactions

Measuring Information Storage



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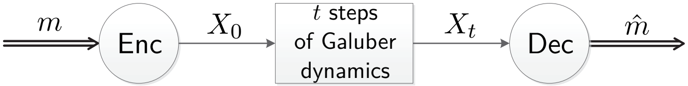

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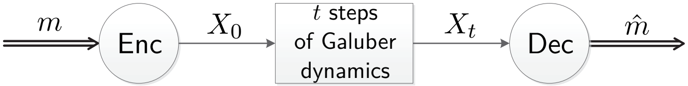

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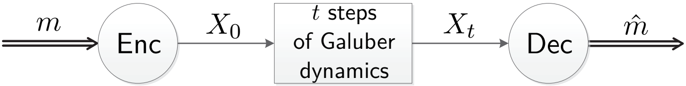

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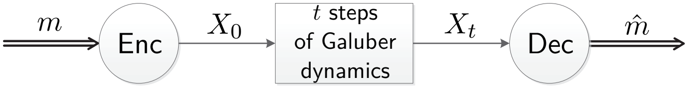

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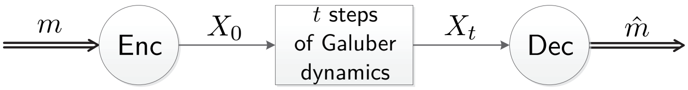

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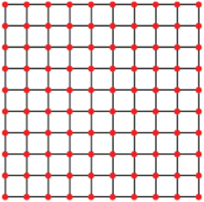

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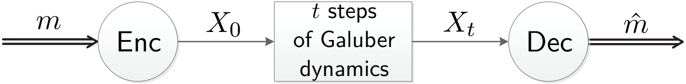

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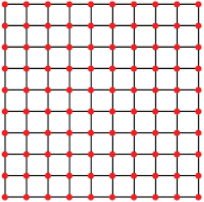

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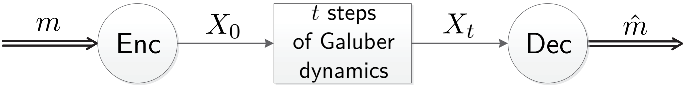

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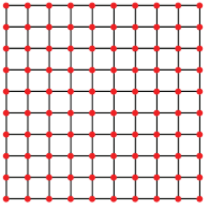

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⊛ **Cold:** Can interactions (memory) help?

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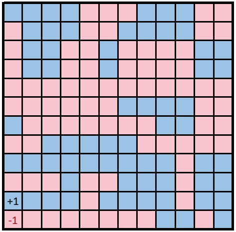

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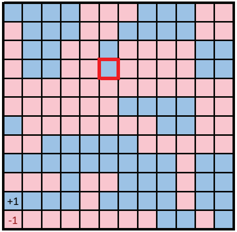

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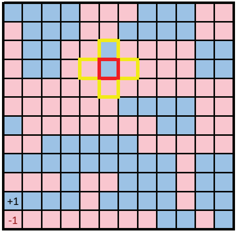

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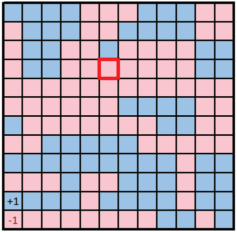

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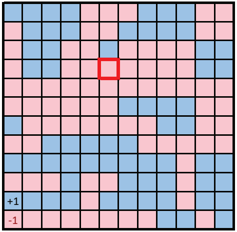

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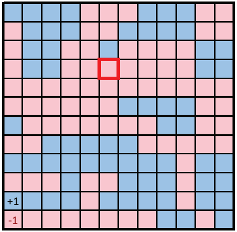

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- **Converse:** See above.
- **Achievability:** Linear codes (Gilbert-Varshamov)

Q1: What (if anything) can be stored for infinite time?

Storing for Infinite Time

Theorem (G.-Bresler-Polyanskiy'19)

For the zero-temp. SIM on $\sqrt{n} \times \sqrt{n}$ *grid* $\quad I_n(\infty) := \lim_{t\to\infty} I_n(t) = \Theta(\sqrt{n})$

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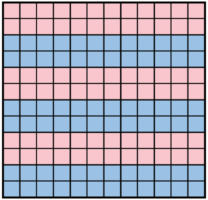

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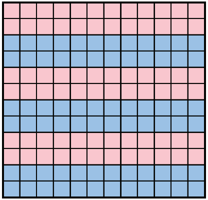

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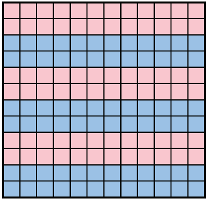

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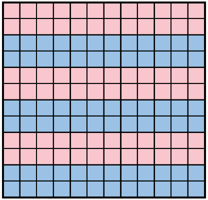

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Q2: Can we do better than $\sqrt{n}$ for finite superlinear t?

Storing $\omega(\sqrt{n})$ Bits for Superlinear Time

Theorem (G.-Bresler-Polyanskiy'19)

Let $a(n) = o(n)$*. Then* $\exists c > 0$ *s.t.* $I_n(t) = \Omega\left(\frac{n}{a(n)}\right), \ \forall t \leq c \cdot a(n) \cdot n$.

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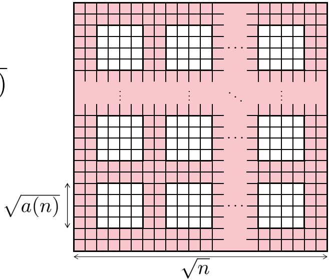

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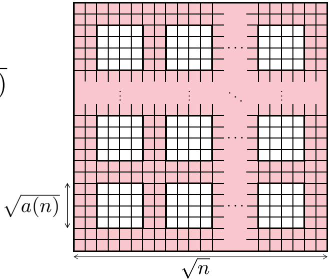

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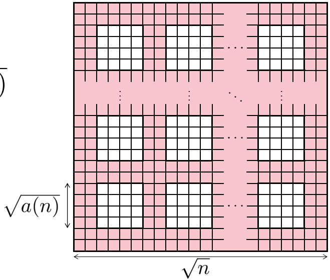

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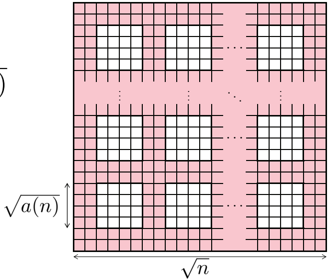

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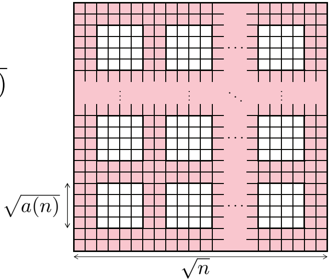

- **Continuous-Time:** Updates according to i.i.d. Poiss$(1/n)$ clocks.

$$I_n(t) \approx I_n^{(c)}((1+o(1))t), \qquad t \sim \text{suplog}(n)$$

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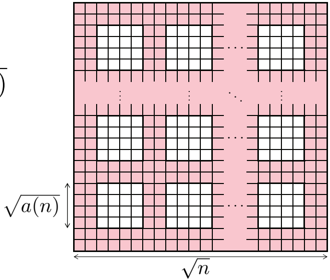

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$\Longrightarrow$ Non-interacting portions are independent.

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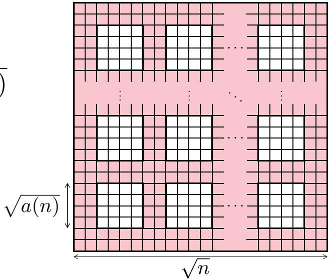

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- **Tensorization:** $I_n^{(c)}(t) \geq K \cdot \max_{p_1} I\left(\left[X_0^{(c)}\right]_1; \left[X_t^{(c)}\right]_1\right)$

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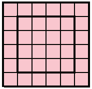

Subsquare:

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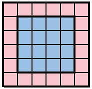

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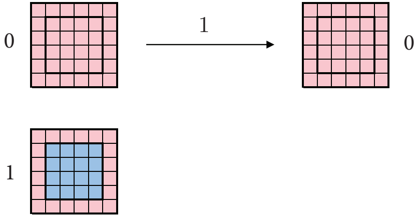

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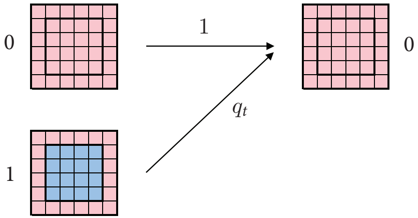

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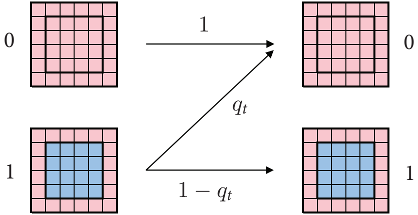

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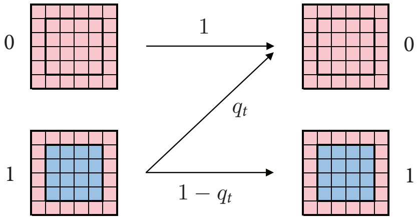

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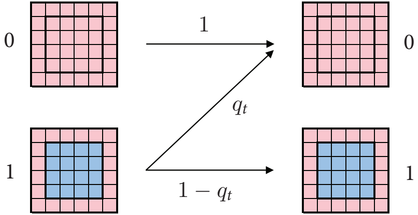

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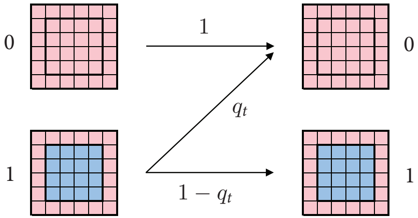

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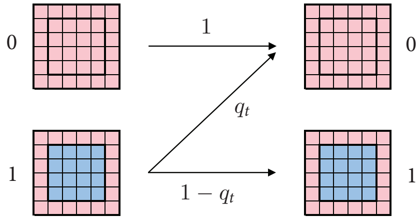

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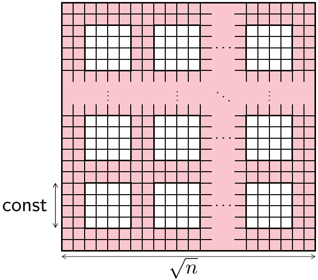

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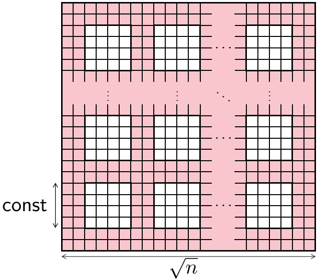

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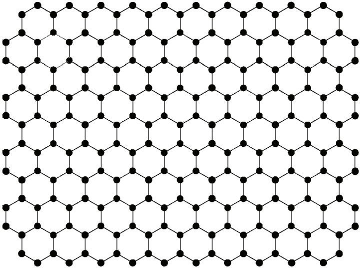

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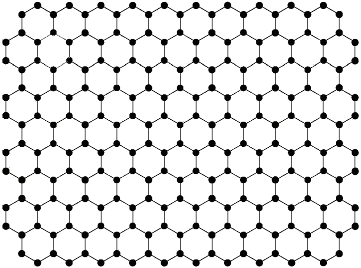

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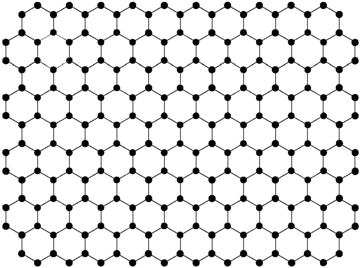

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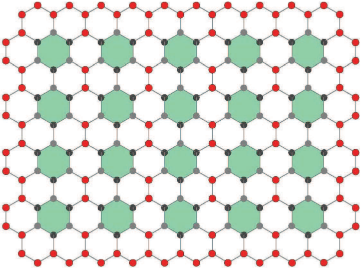

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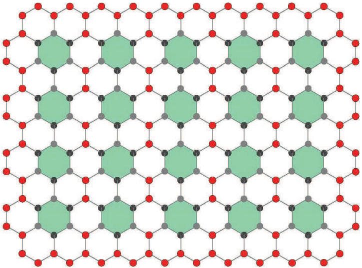

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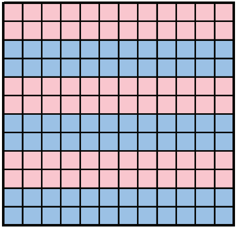

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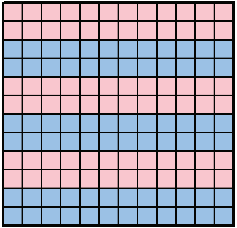

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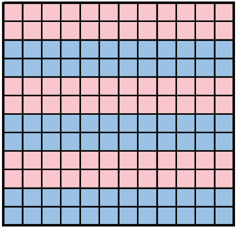

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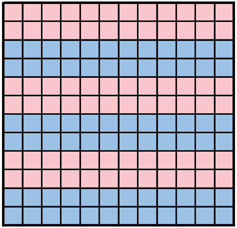

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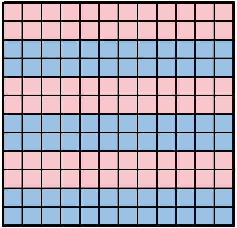

Reduction to Single Stripe Analysis

- **Denote:** $t_f \triangleq e^{c\beta}$; $X_t^{(j)} \triangleq X_t^{(j)}\Big|_{\text{Stripe } j}$; $X_t^{[j]} \triangleq \big(X_t^{(k)}\big)_{k=1}^{j}$

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$\Longrightarrow$ Suffices to analyze $\mathbb{P}$(More than half stripe flipped)

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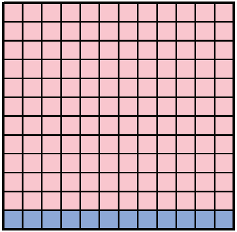

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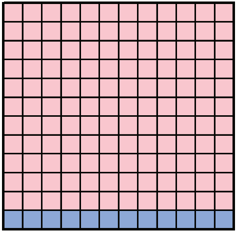

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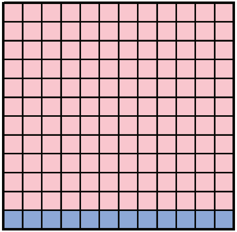

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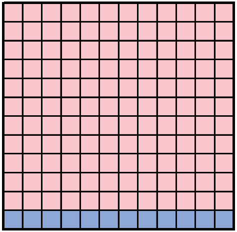

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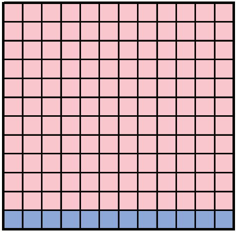

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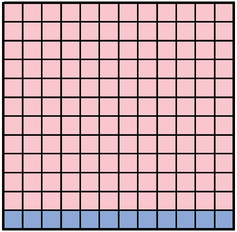

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Theorem (Goldfeld-Bresler-Polyanskiy'19)

Fix any $c, C \in (0,1)$. For β and n sufficiently large, we have

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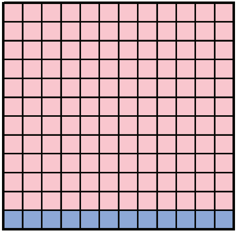

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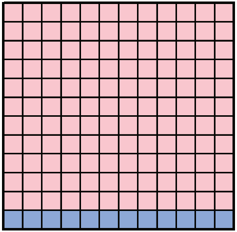

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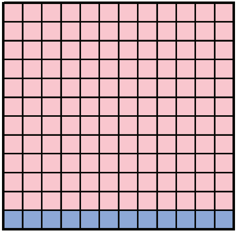

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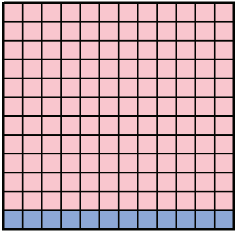

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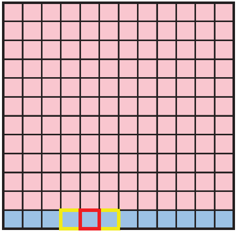

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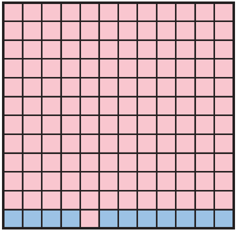

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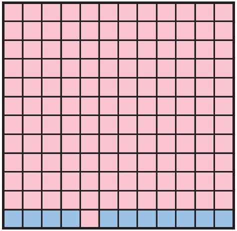

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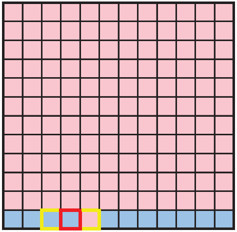

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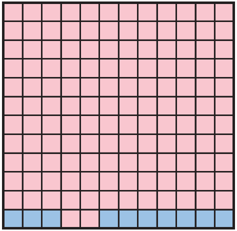

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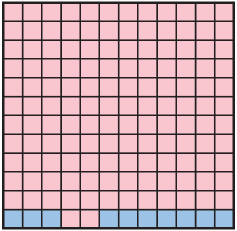

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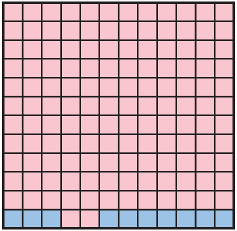

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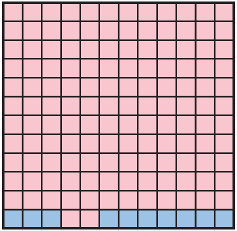

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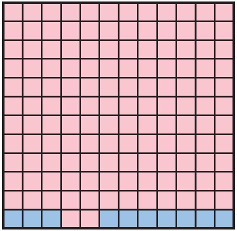

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$\Longrightarrow$ Dominate $\{X_t\}_t$ by a phase-separated dynamics

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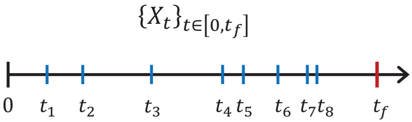

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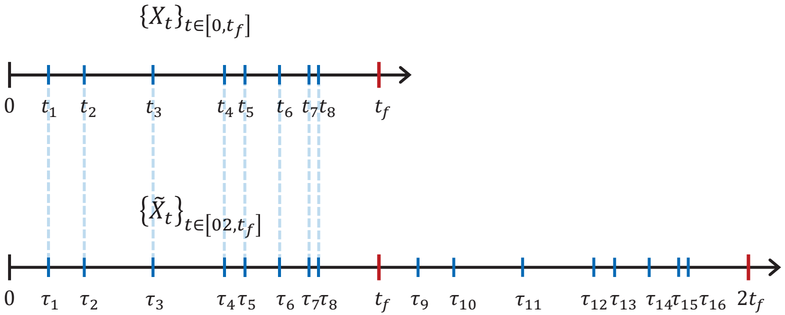

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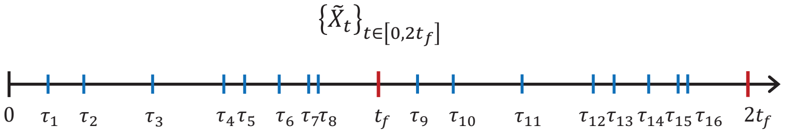

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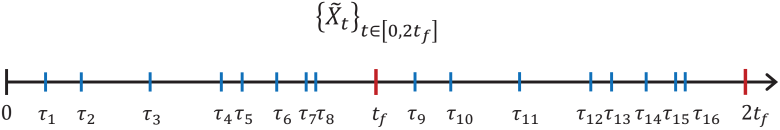

Blocking Rule:

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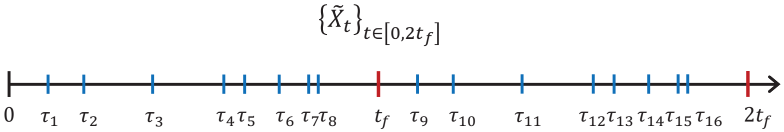

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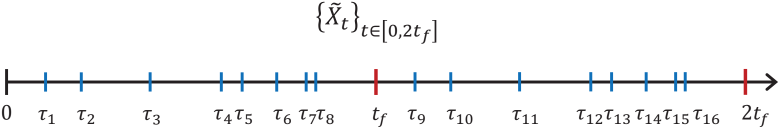

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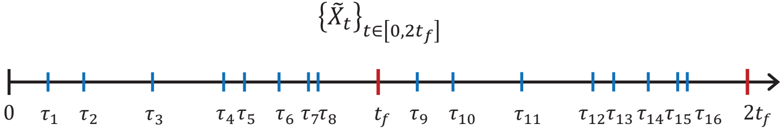

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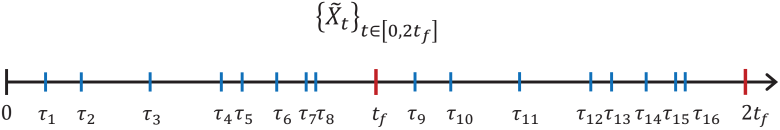

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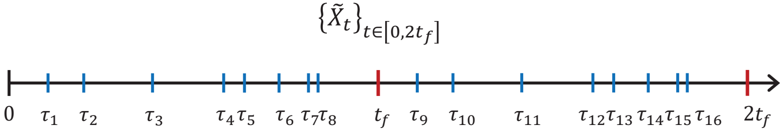

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- Erosion flips in $\{X_t\}_{t\in[0,t_f]} \implies$ Erosion flips in $\{\tilde{X}_t\}_{t\in[t_f,2t_f]}$

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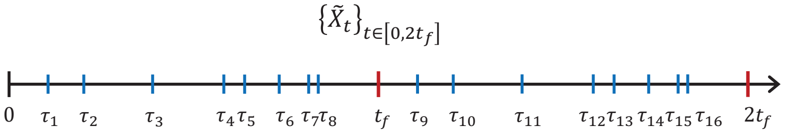

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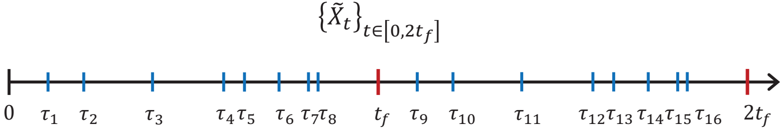
⊛ Adjust Poisson clock rates of $\{\tilde{X}_t\}_{t\in[0,2t_f]}$ to neighborhoods

Observations:

- Erosion flips in $\{X_t\}_{t\in[0,t_f]} \implies$ Erosion flips in $\{\tilde{X}_t\}_{t\in[t_f,2t_f]}$
- Erosion flip rates in $\{\tilde{X}_t\}_{t\in[t_f,2t_f]}$ are faster.

Single Stripe Case: Phase-Separated Dynamics (2)



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$\implies$ New dynamics is a speedup: $\mathbb{E}N^{(+)}(t_f) \ge \mathbb{E}\tilde{N}^{(+)}(2t_f)$

Single Stripe Case: Phase-Separated Dynamics (3)

Sprinkle Analysis $[0, t_f]$: Ends w/ runs of '+'s separated by '-' sprinkles

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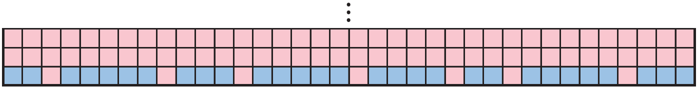

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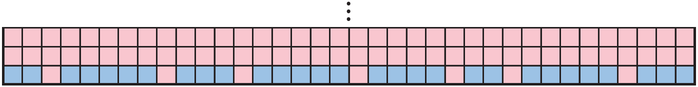

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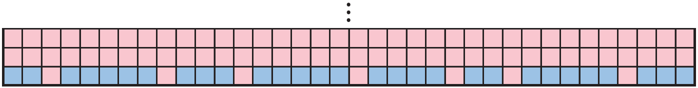

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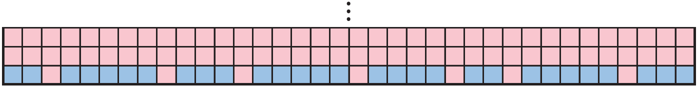

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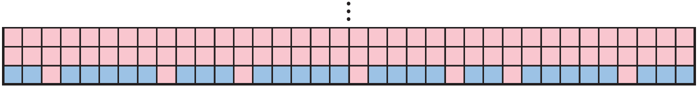

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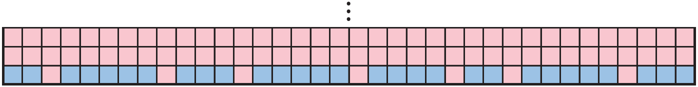

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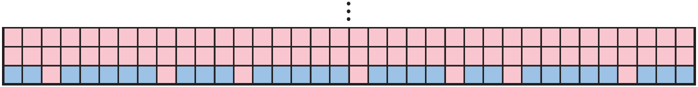

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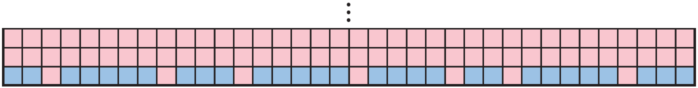

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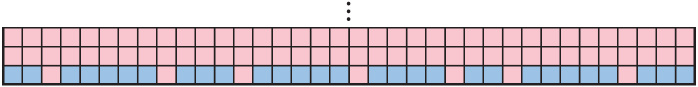

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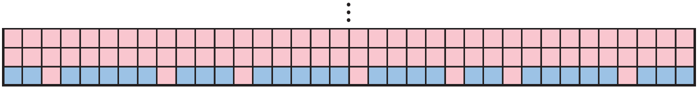

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- $\left\{\text{Half contig eaten in } t_f \text{ time}\right\} = \left\{\sum_{i=1}^{\ell_\beta/2} \mathsf{Exp}(\phi_\beta) \leq t_f\right\}$
- Show latter probability is small and conclude proof