# Smooth Wasserstein Distance: Metric Structure and Statistical Efficiency

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Workshop on Coding, Cooperation, and Security in Modern Communication Networks

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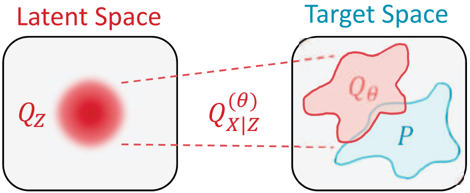

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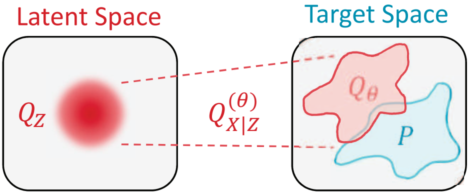


**Minimum Distance Estimation:** Solve $\boxed{\theta^\star \in \underset{\theta}{\operatorname{argmin}}\, \delta(P, Q_\theta)}$

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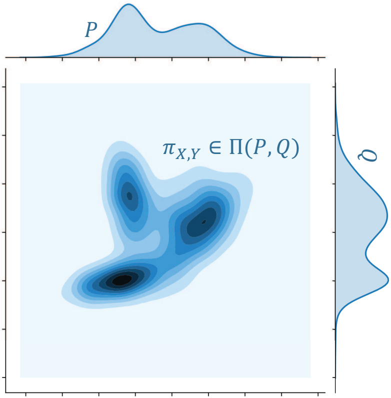

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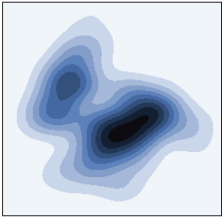

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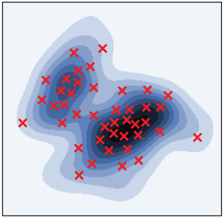

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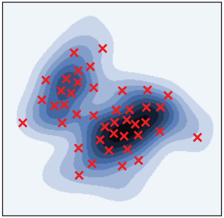

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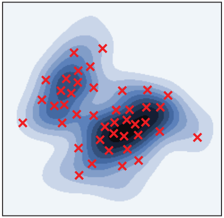

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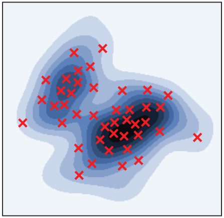

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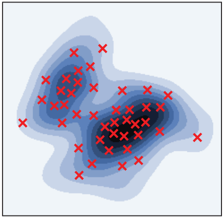

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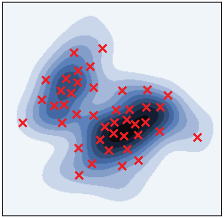

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For $\sigma \geq 0$, the smooth 1-Wasserstein distance between $P$ and $Q$ is

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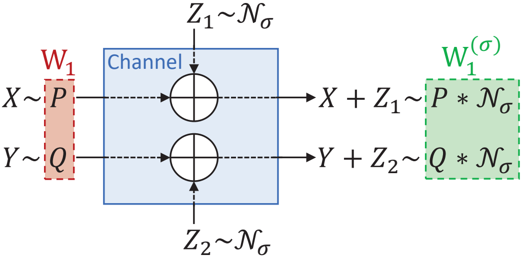

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$$\mathsf{W}_1^{(\sigma)}(P,Q) = \sup_{f \in \mathsf{Lip}_1(\mathbb{R}^d)} \mathbb{E}_P[f * \varphi_\sigma] - \mathbb{E}_Q[f * \varphi_\sigma] = \sup_{g \in \mathcal{F}_\sigma} \mathbb{E}_P[g] - \mathbb{E}_Q[g]$$

$\Longrightarrow$ **Emp. $\mathbf{W}_1^{(\sigma)}$:**

$$\boxed{\sqrt{n}\mathsf{W}_1^{(\sigma)}(P_n, P) = \sqrt{n} \sup_{g \in \mathcal{F}_\sigma} \underbrace{\frac{1}{n}\sum_{i=1}^n g(X_i) - \mathbb{E}_P g}_{\triangleq (P_n - P)g}}$$

⊛ **Objective:** $\mathbb{G}_n = \left(\sqrt{n}(P_n - P)g\right)_{g \in \mathcal{F}_\sigma} \xrightarrow{w}$ GP $\iff$ $\mathcal{F}_\sigma$ is $P-$Donsker

2. **[van der Vaart'96]:** $K \subseteq \mathbb{R}^d$ bdd & convex: $\mathcal{F}_\sigma\Big|_{\mathcal{K}} \subseteq C_M^N(\mathcal{K}),\ N > \frac{d}{2}$

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3. **Continuous Mapping Theorem:** $\boxed{\sqrt{n}\mathsf{W}_1^{(\sigma)}(P_n, P) \xrightarrow{d} \sup_{g \in \mathcal{F}_\sigma} G_P(g)}$

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- **Advanced:** $\sup\limits_{f \in \mathsf{Lip}_1} \mathbb{E}(f * \varphi_\sigma)(X) - \mathbb{E}(f * \varphi_\sigma)(Y)$ & implement '$*$'

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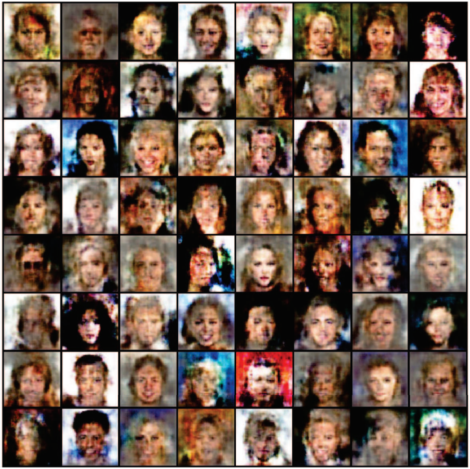
Baseline W-GAN ($\sigma = 0$)



Smooth W-GAN ($\sigma = 0.1$)

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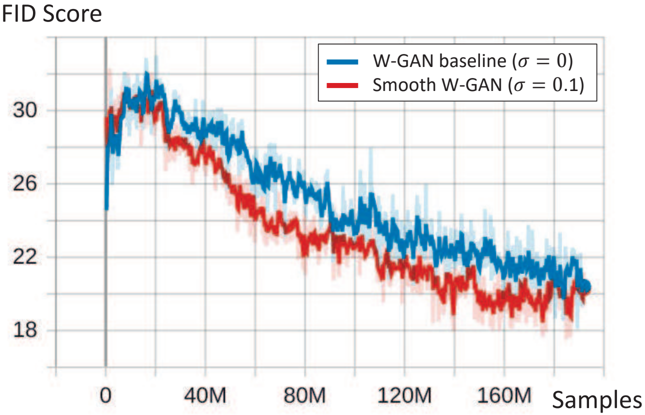


⊛ FID = Fréchet Inception Distance

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