Smooth Wasserstein Distance: Metric Structure and Statistical Efficiency

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July 2020

Implicit (Latent Variable) Generative Models

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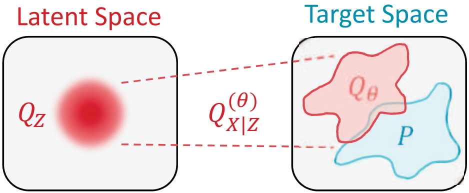

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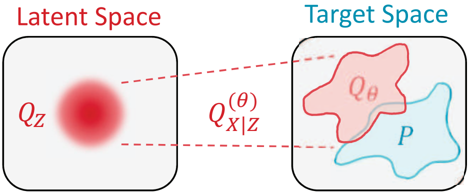

Minimum Distance Estimation: Solve $\theta^\star \in \underset{\theta}{\operatorname{argmin}}\, \delta(P, Q_\theta)$

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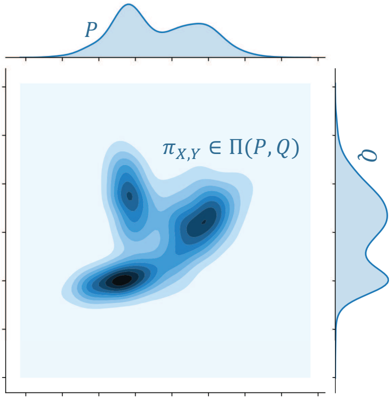

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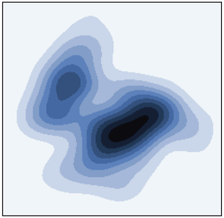

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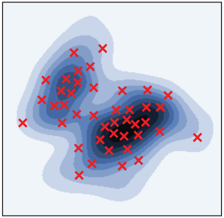

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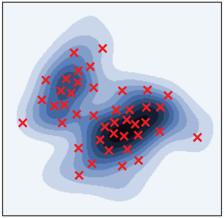

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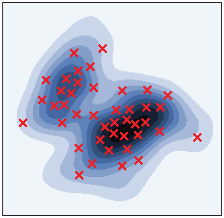

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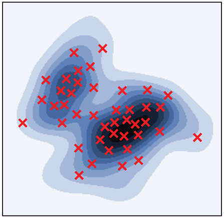

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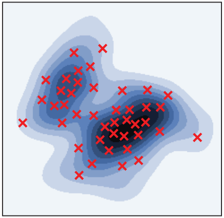

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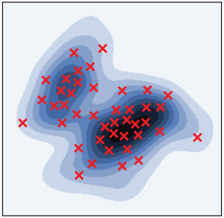

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⊛ **Question:** Can we find a new W_1-like distance that alleviates CoD?

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For $\sigma \geq 0$, the smooth 1-Wasserstein distance between P and Q is

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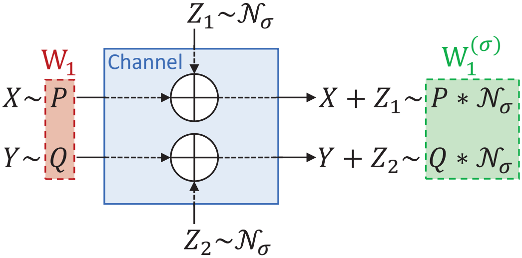

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Let $P_n, P \in \mathcal{P}_1(\mathbb{R}^d)$, $n \geq 1$. *Then:* $\mathsf{W}_1^{(\sigma)}(P_n, P) \to 0$ *iff* $\mathsf{W}_1(P_n, P) \to 0$

Smooth 1-Wasserstein – Metric Structure

High Level: $\mathsf{W}_1^{(\sigma)}$ inherits the metric structure of 1-Wasserstein

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$\left(\mathcal{P}_1(\mathbb{R}^d), \mathsf{W}_1^{(\sigma)}\right)$ *is metric space,* $\forall \sigma \geq 0$ *(and* $\mathsf{W}_1^{(\sigma)}$ *metrizes weak conv.).*

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⊛ $\mathsf{W}_1^{(\sigma)}$ and W_1 induce same topology

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For any $d \geq 1$, $\sigma > 0$ and sub-Gaussian P: $\mathbb{E}\mathsf{W}_1^{(\sigma)}(P_n, P) \lesssim n^{-\frac{1}{2}}$

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Under same assumptions: $\sqrt{n}\mathsf{W}_1^{(\sigma)}(P_n, P) \xrightarrow{d} \sup_{f \in \mathsf{Lip}_1(\mathbb{R}^d)} G_P^{(\sigma)}(f)$,
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$$\sqrt{n}\mathsf{W}_1^{(\sigma)}(P_n, P) = \sqrt{n} \sup_{g \in \mathcal{F}_\sigma} \underbrace{\frac{1}{n}\sum_{i=1}^n g(X_i) - \mathbb{E}_P g}_{\triangleq (P_n - P)g}$$

⊛ **Objective:** $\mathbb{G}_n = \left(\sqrt{n}(P_n - P)g\right)_{g \in \mathcal{F}_\sigma} \xrightarrow{w} \mathsf{GP} \iff \mathcal{F}_\sigma$ is $P-$Donsker

2. **[van der Vaart'96]:** $K \subseteq \mathbb{R}^d$ bdd & convex: $\mathcal{F}_\sigma\Big|_{\mathcal{K}} \subseteq C_M^N(\mathcal{K}),\ N > \frac{d}{2}$

$\Longrightarrow$ **$\mathcal{F}_\sigma$ is P-Donsker:** $\mathbb{G}_n \xrightarrow{w} G_P$ (Gaussian process)

3. **Continuous Mapping Theorem:** $\sqrt{n}\mathsf{W}_1^{(\sigma)}(P_n, P) \xrightarrow{d} \sup_{g \in \mathcal{F}_\sigma} G_P(g)$

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Smooth Wasserstein GAN: Initial Empirical Results

Setup: f parametrized by 2-layer NN & conv. computed in closed form

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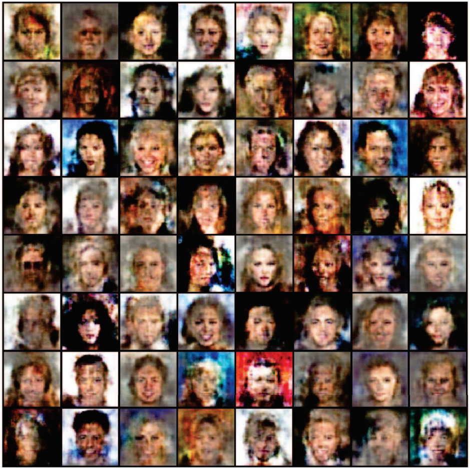
Baseline W-GAN ($\sigma = 0$)



Smooth W-GAN ($\sigma = 0.1$)

Smooth Wasserstein GAN: Initial Empirical Results

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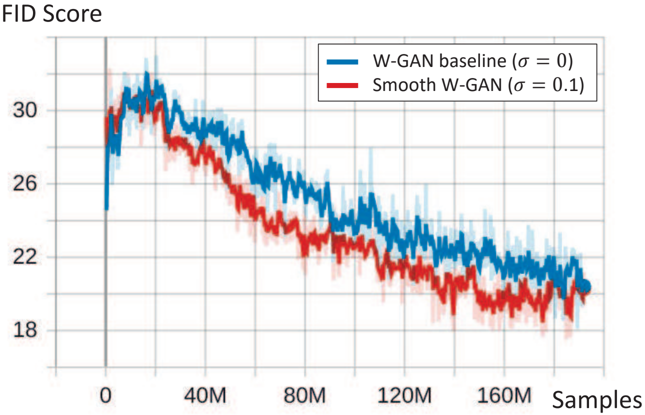

⊛ FID = Fréchet Inception Distance

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