Smooth Wasserstein Distance: Metric Structure and Statistical Efficiency

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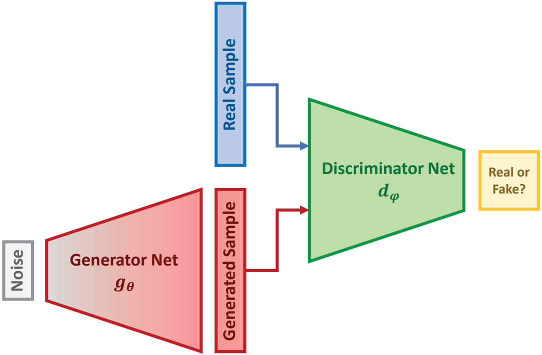

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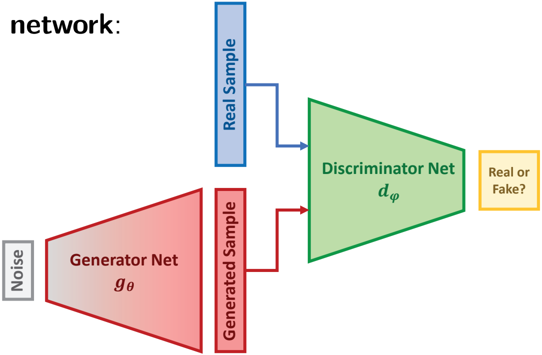

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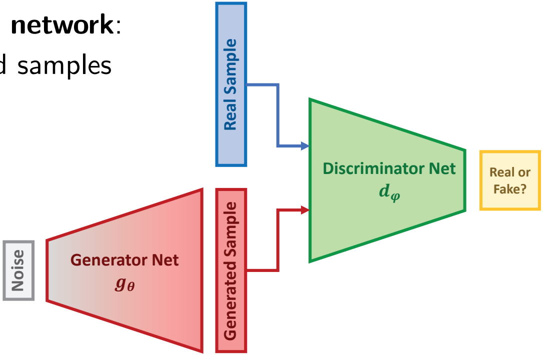

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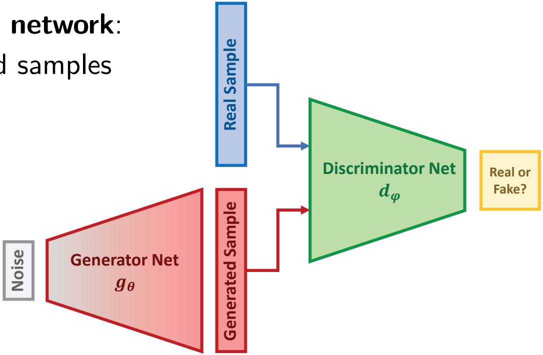

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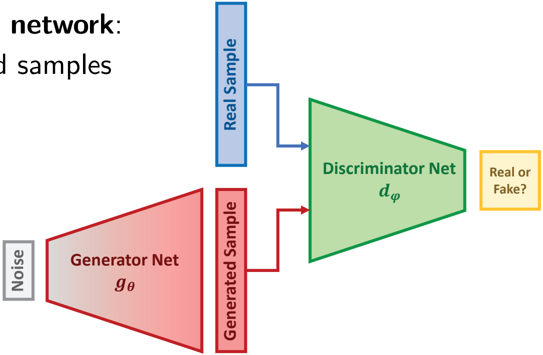

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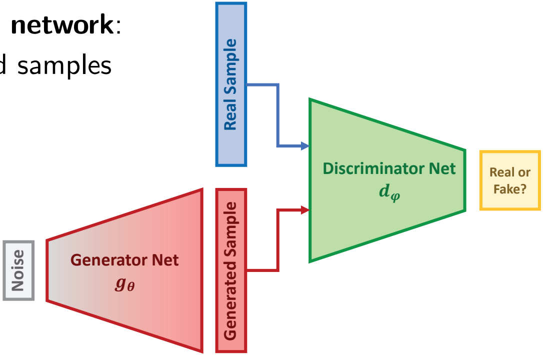

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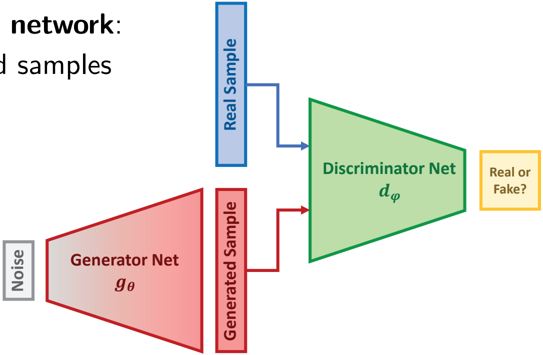

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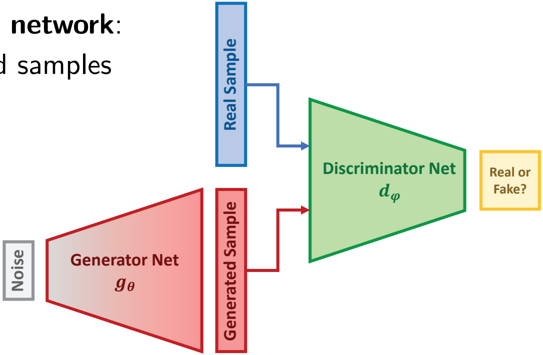

Question:

How to quantify $Q_\theta \approx P$?

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Quantification: Via statistical divergence

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⊛ **Pros:** Metric on $\mathcal{P}_1(\mathbb{R}^d)$ & Robust to supp. mismatch $\mathsf{W}_1(P, Q) < \infty$

Duality & Wasserstein GAN

Kantorovich-Rubinstein Duality:

$$\mathsf{W}_1(P,Q) = \sup_{f \in \mathsf{Lip}_1(\mathbb{R}^d)} \mathbb{E}_P f(X) - \mathbb{E}_Q f(Y)$$

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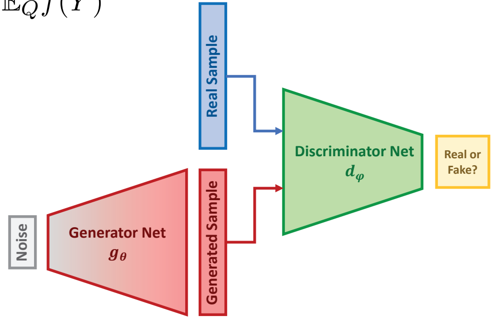

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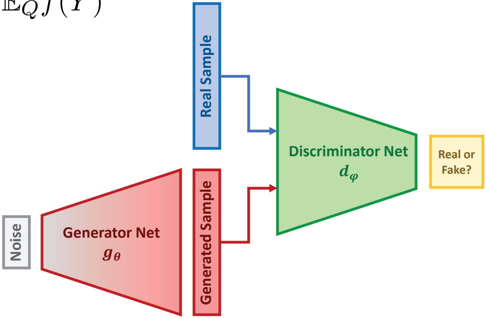

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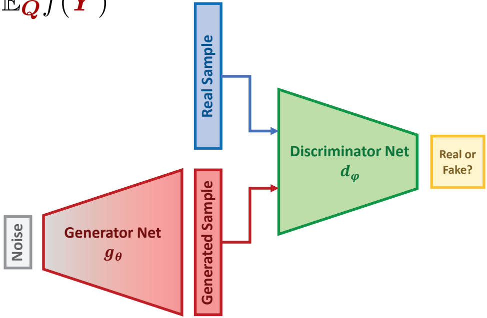

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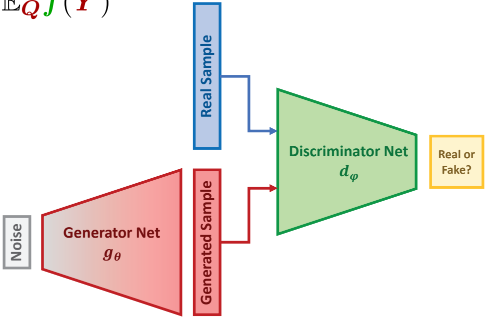

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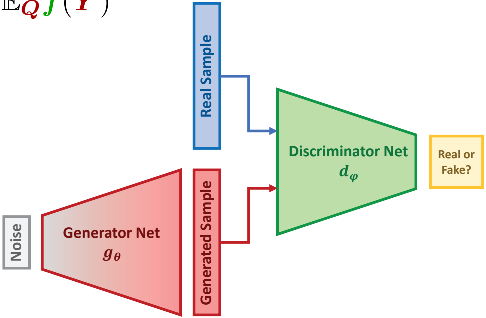

$$\Longrightarrow \quad \boxed{\inf_\theta \mathsf{W}_1(P, Q_\theta) \cong \inf_\theta \sup_{\varphi:\ d_\varphi \in \mathsf{Lip}_1(\mathbb{R}^d)} \mathbb{E} d_\varphi(X) - \mathbb{E} d_\varphi(g_\theta(Z))}$$

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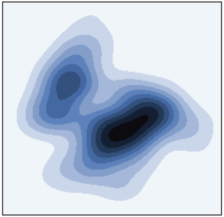

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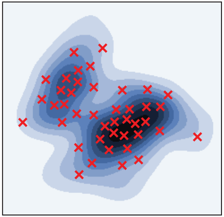

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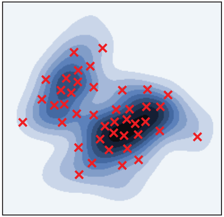

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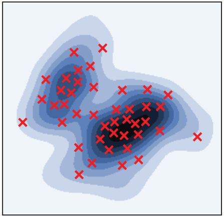

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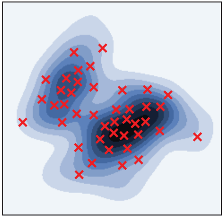

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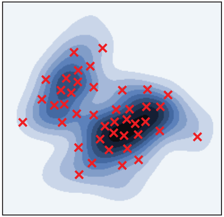

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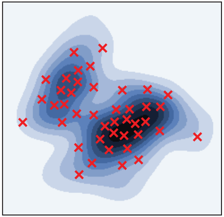

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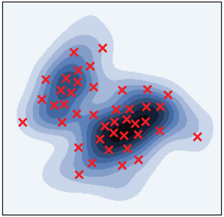

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⊛ **Goal:** Define a new metric that alleviates CoD

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Definition (ZG-Greenewald'19)

For $\sigma \geq 0$, the smooth 1-Wasserstein distance between P and Q is

$$\mathsf{W}_1^{(\sigma)}(P,Q) \triangleq \mathsf{W}_1(P * \mathcal{N}_\sigma, Q * \mathcal{N}_\sigma),$$

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Retain KR Duality: $\mathsf{W}_1^{(\sigma)}$ is W_1 but between convolved distributions:

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